

A Thesis Submitted for the Degree of PhD at the University of Warwick

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**Dynamical Behaviour of Digital Filters subject to
2's Complement Arithmetic Nonlinearity**

(Two Volumes)

Volume I

by

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Thesis

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for the degree of

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Declaration

No material contained in this thesis has been used before, in accordance with Section 3.5 of the "Guide to Examinations for Higher Degrees by Research" (Graduate School, University of Warwick). Prior to submission, no part of it has been submitted for publication. The content of this thesis is the original work of the author.

Synopsis

The thesis investigates a two-dimensional nonlinear map describing a second-order direct form digital filter constrained to operate with 2's complement arithmetic. Attention is primarily directed towards the lossless case in which the filter parameters are on the lower boundary of the linear stability region. An effective tool for studying the dynamics of the map is the association of an infinite symbolic sequence with each orbit. Identification of the admissible periodic sequences is a fundamental theme.

A comprehensive answer is given for when string fragments $+0 \dots 0+$, and related patterns, appear within some admissible sequence. Infinite families of admissible periodic sequences are exhibited as analytic solutions to the trigonometric formulation of Chua and Lin's admissibility criterion. The role of the filter parameter $a = 2 \cos \theta$ in determining admissibility is emphasised. Examples demonstrate that a periodic sequence may be admissible over several disjoint parameter intervals.

Strategies are developed that permit powerful computer searches for admissible periodic sequences. Tables are included listing the periodic sequences admissible for fixed values of the parameter, up to period 200. Parameter values for the tabulation progress by steps of 0.05 across the entire range $(0, \pi)$. A complete list, covering all parameter values, of the admissible periodic sequences up to period 20 is presented; associated with each sequence is its interval of admissibility.

A classification is undertaken of all periodic sequences with an interval of admissibility having left end-point $\theta = 0$. For odd period sequences it is definitive; in the even case a partial classification is achieved, definitive when the period is twice a prime. An investigation of comparable scope is presented for admissibility around $\theta = \frac{\pi}{2}$.

The topological dynamics of a one-dimensional reduction of the filter map is explored. Orbits are attracted to an invariant set, itself a disjoint union of intervals, that becomes increasingly fragmented as the parameter a approaches 1.

Chapter 1

Introduction

The purpose of this introduction is to explain in general terms the subject of this thesis, to describe where the work began, and to outline the scope of the original contributions as set forth in the later chapters. The style adopted here is non-technical, the formal presentation commences in Chapter 2.

Our principal object of study is a nonlinear map associated with the operation of a digital filter. The map describes the state to state transitions when the filter is realistically implemented, i.e. constrained to work with some real-world, imperfect, arithmetic system. In this situation the filter operation may depart from the idealised linear model employed widely for digital signal processing; nonlinear effects can dominate and the actual behaviour of the filter may be altogether different. The nonlinear map is a direct characterisation of the filter operation and values the map assumes are essentially the output signal from the filter; the relation is more intimate than that of a mathematical model simulating some physical system. Our concern therefore is with a mathematical study of the dynamics of the nonlinear map; questions of compatibility between a mathematical model and the modelled system do not arise. This accounts for the high proportion of mathematical theory to be found in the thesis, which perhaps at first is surprising given the practical context of the study; in this respect our work does not differ from that of others in the field. The behaviour of the nonlinear map in general, and many of the particular phenomena that one can easily exhibit through computer plotting, are not well understood.

In specific terms the action of the map can be visualised as a geometric transformation. The process is twofold, a linear deformation involving a clockwise rotation by $\frac{\pi}{2}$ and a shear in the vertical direction, followed by a nonlinear and discontinuous cut-and-paste operation. Figure 1.1 shows what is intended.

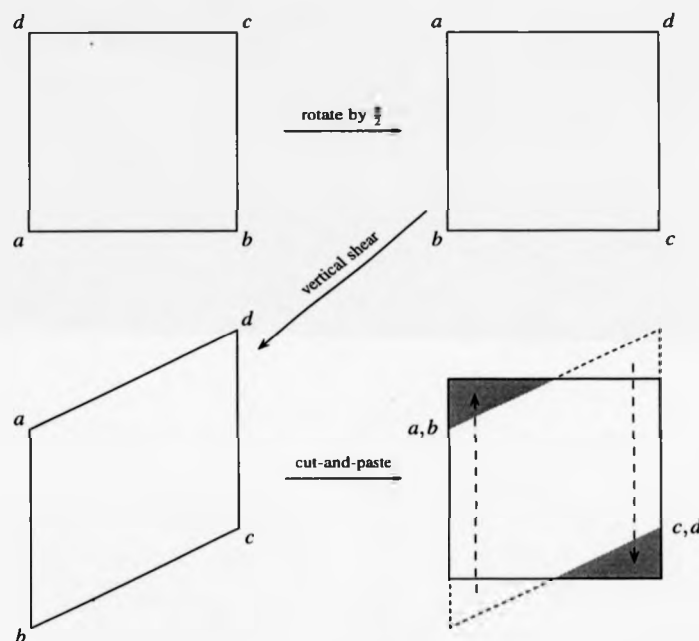


Figure 1.1: The nonlinear map as a three stage geometric transformation.

The transformation has the effect of mapping the original square back onto itself. Reverting to the digital filter connection, the coordinates of a point in this square correspond to the current state of the filter, i.e. the values of the output signal, now and at the previous time step. The action of the map, taking a point from one location in the square to another, executes the time-evolution of the state of the filter, the output signal is simply one of the coordinate projections. The orbit of a point, comprising its iterates under the map, represents the history of the output from the filter.

Of prime interest to us are certain symbolic sequences associated with the dynamical evolution of the system. One of the three symbols $+$, 0 or $-$ is used to record what happens, during the cut-and-paste step, to a point when it is mapped : 0 if it is untouched, $-$ if it must be translated downwards and $+$ if it must be translated upwards. To the whole orbit of a point is assigned an infinite sequence of these symbols. A classification of the orbits that arise can be linked to the various types of sequences. In particular orbits associated with periodic sequences, though not necessarily periodic themselves, form an interesting study, significant for the understanding of the dynamics of the map.

We make all of the foregoing more precise, and justify the assertions, in Chapter 2. That chapter commences with an account of the basic theory, our main source for this is Chua and Lin (1988), but we have adapted many of their ideas to fit more smoothly with our subsequent investigations. Otherwise Chapter 2 takes up in more detail the practical origins of this subject and sets our particular study in the wider context of the work of other authors on the nonlinear maps associated with digital filters.

Not every symbolic sequence arises in association with an orbit, and posed in the literature as a significant problem, is the task of describing those that do. Of particular interest in this connection is the case of the periodic sequences. The published empirical evidence gathered about these sequences is extremely limited, and a key objective of the early part of this thesis is to develop computational strategies permitting a much more extensive search. Chapter 4 describes the computational approach that we have adopted and the results of our searches are summarised in the tables that form the second half of Volume II.

What was revealed very clearly by the computer searches is the great multiplicity of form for the periodic sequences. One consequence is that our tabulation is necessarily extensive, and could not sustain curtailment without seriously degrading its value. Indeed we do feel that the lists we have been able to present in the space available only go some way to reporting the variety of information that our computer searches uncovered, and on occasions we need to include in the text isolated results drawn from more exhaustive searches.

The ideas underlying the computer algorithms build upon material developed in Chapter 3. One way of analysing the nature of the symbolic sequences that arise in association with orbits is to decide the question of whether or not they contain particular types of substring. For certain elemental families of strings, we are able to resolve the issue completely, both unifying all the earlier results in the literature and offering a new and more general means for addressing this kind of issue.

In Chapters 5 and 6 we follow up the evidence gleaned from the computer searches to begin a theoretical study of the apparent structure present amongst the sequences. A feature of the nonlinear map that we have not yet mentioned is the presence of a parameter which varies the extent of the shear and so determines the amount of the square shifted by the cut-and-paste. The lists of sequences in Volume II are tabulated according to values of this parameter, which had not previously been done. At certain significant parameter values we observed regular families of sequences and their analytical investigation is the subject of Chapter 5. Motivated by the conclusions reached, we investigate in Chapter 6, an interesting family of sequences amenable to the methods that proved effective in Chapter 5. Although simple in form, the sequences of this family exhibit surprising and subtle properties which makes them difficult objects to study computationally: no member of the family had been encountered amongst the results of our computer searches.

There is a shift of emphasis in Chapters 7 and 8. After the study of individual families of sequences, we now turn to the much more difficult task of classifying all the periodic sequences associated with the nonlinear map. Our contribution is to achieve a classification near to both ends of the parameter range, where it is notable that the sequences cluster. One feature of the work is that we are able to give a collective description for all of these sequences. Chapter 9 develops the consequences of what is accomplished in Chapters 7 and 8; knowledge of the sequences gives back information about the dynamics of the map, and we give an application of this kind.

Some further investigations conducted during the period of research for this thesis are summarised briefly in Appendix A.

Appended for reference is a short table of trigonometric formulae which are used intensively within some sections of the thesis.

Addition Formulae

$$\begin{aligned}\sin A + \sin B &= 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) & \cos A + \cos B &= 2 \cos \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B) \\ \sin A - \sin B &= 2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B) & \cos A - \cos B &= -2 \sin \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)\end{aligned}$$

Product Formulae

$$\begin{aligned}2 \sin A \cos B &= \sin(A+B) + \sin(A-B) \\ 2 \sin A \sin B &= \cos(A-B) - \cos(A+B) \\ 2 \cos A \cos B &= \cos(A+B) + \cos(A-B)\end{aligned}$$

Shift of Origin

$$\begin{aligned}\sin\left(\theta + \frac{\pi}{2}\right) &= \cos \theta & \cos\left(\theta + \frac{\pi}{2}\right) &= -\sin \theta & \sin(k\pi + \theta) &= (-1)^k \sin \theta \\ \sin\left(\frac{\pi}{2} - \theta\right) &= \cos \theta & \cos\left(\frac{\pi}{2} - \theta\right) &= \sin \theta & \cos(k\pi + \theta) &= (-1)^k \cos \theta\end{aligned}$$

Chapter 2

A Nonlinear Map Arising in the Study of Digital Filters

This chapter sets the scene for the material in the main body of the thesis, both in detailing the required mathematical background and in providing a context for the later work by relating it to the existing literature.

The content is organised into four sections. Section 2.1 presents an account of the essential theory, from the mathematical point of view, introducing the nonlinear map whose investigation is at the core of our study, discussing its associated dynamics, and showing the crucial role played by the admissible periodic sequences in explaining the orbits.

In Section 2.2 a historical perspective is provided, explaining how the mathematical study arose out of an observation in the late 1960's of overflow oscillations in digital filters. Section 2.3 shows how our particular nonlinear map fits naturally into a wider family of maps, themselves modelling a digital filter under its various operating regimes. We survey the studies reported and briefly indicate the conclusions reached and the relevant methods.

The chapter concludes with Section 2.4, in which we bring together some elementary (but useful) properties of admissible periodic sequences, that otherwise are scattered through the literature and reported in a variety of inconsistent ways.

2.1 Introduction to the underlying dynamical system

2.1.1 Definition of the nonlinear map

Our fundamental concern is with the dynamics of the two-dimensional nonlinear map F , defined by

$$F(\mathbf{x}) = A\mathbf{x} + s(\mathbf{x})\mathbf{b}, \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad (2.1)$$

where A is the matrix $\begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$ dependent on the real parameter a which is restricted so that $-2 < a < 2$, \mathbf{b} is the constant vector $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, and s is a function which is responsible for the nonlinear behaviour of the map and is specified by

$$s(\mathbf{x}) = \begin{cases} -1 & \text{if } -x_1 + ax_2 \geq 1, \\ 1 & \text{if } -x_1 + ax_2 < -1, \\ 0 & \text{otherwise.} \end{cases} \quad (2.2)$$

We study F on the square domain $I^2 = [-1, 1) \times [-1, 1)$, within \mathbb{R}^2 , on which F is bijective. The action of the function s is to pull back, by means of a vertical translation, those points sent out of I^2 by the linear component $\mathbf{x} \mapsto A\mathbf{x}$ of F . The map F models the state-space transitions of a second-order digital filter; the relevance of F to the subject of digital filters and its role in modelling a specific filter realisation is treated in Section 2.2.

We wish to study the iterated effect of F on points within the domain I^2 . Numerical simulation of these orbits by computer plotting indicates that the manner in which they fill out the state-space can be extremely complex. Some impression of the subtle behaviour to expect may be gained from Figure 2.1, which is a representation of a single orbit in the case when $a = 1.5$. To investigate the dynamical system associated with the map F , it is advantageous to decompose the state-space I^2 into subsets throughout each of which the map F exhibits much the same behaviour. The means whereby this is achieved is to associate to each point \mathbf{x} in I^2 an infinite *symbolic sequence* $s_0s_1s_2s_3\dots$, consisting of the

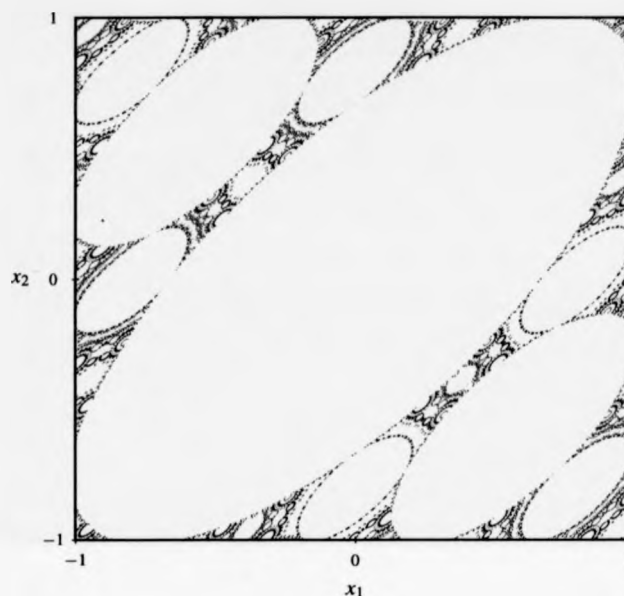


Figure 2.1: The complex nature of the orbits of the map F : the plot shows a single orbit with initial point $(x_1, x_2) = (0.1, -0.96)$ when $a = 1.5$.

succession of values taken by the function s when the point x is repeatedly iterated under F ; we say that the point x gives rise to the sequence $s_0 s_1 s_2 s_3 \dots$. By way of notation we represent each digit in the sequence by $-$, 0 and $+$, according as s takes the values -1 , 0 or 1 ; so that, for example, when the parameter a takes the value $\frac{1}{2}$, the sequence associated to the point $(\frac{9}{10}, -\frac{3}{10})$ is found by iteration to commence:

$+00+-+00-00-++00+00+-+00-00-++00+00+-+00-00\dots$

It is important to note that, with this association of points in I^2 to symbolic sequences, the correspondence between points in the set I^2 and the collection of all three-symbol sequences is neither one-to-one nor onto. A single symbolic sequence may arise from distinct points within I^2 , and conversely not every symbolic sequence need arise from a point in I^2 .

Indeed, in relation to the latter circumstance, a key concept for us is the notion of an *admissible sequence*, which we define to be a symbolic sequence that does arise in association with the iteration, under F , of some point within I^2 .

In this thesis our concern is primarily with the points which give rise to *periodic* symbolic sequences. These points are important since they comprise a large proportion of the domain I^2 ; indeed the set of points in I^2 that corresponds to any single admissible periodic sequence has positive measure. Moreover, the points which give rise to periodic sequences form an attractive subject to study, because we are able to describe completely the orbit of each, and also to specify precisely the region within I^2 which is the set of points giving rise to some particular periodic sequence. We give an account of all of this, and elaborate the theory, in Section 2.1.3.

Although the relationship between the admissible periodic sequences and the dynamics of the map F is well understood, it is applicable only after an admissible periodic sequence has been identified as such, and in fact very little is known about which periodic sequences are admissible. Accounting for the collection of admissible periodic sequences is a challenging and deep problem central to the understanding of the nature and complexity of the dynamics of the nonlinear map F . In this thesis we present a contribution to the research on this problem, revealing for the first time the rich structure present in the collection of admissible periodic sequences. We are able to uncover many new infinite families of admissible periodic sequences. Our focus is particularly on the dependence between the admissibility of the sequences belonging to these families and the parameter of the nonlinear map.

2.1.2 The identification of strings that generate admissible periodic sequences

We present a result due to Chua and Lin that serves as a foundation and starting point for the study of admissible periodic sequences. It is a criterion to ascertain whether some given periodic sequence is admissible in terms of the locations of the solutions of a related collection of linear equation systems. In the course of this section we introduce notation

and ideas that will be relevant for much of our later work. As an instance, we remark that any periodic symbolic sequence may be specified by a string, of finite length, each of whose digits is one of the symbols $+$, 0 and $-$; the periodic sequence is *generated* from this string by repeated concatenation of copies of it. (In this connection, we note that we refrain from ever using the phrase "admissible string" because of the evident ambiguity. Does it mean a string that can appear as a substring within an admissible sequence, or is it the generator for an admissible periodic sequence? The two notions are different and we will be making use of them both. We encountered some confusion in the existing literature.)

Proposition 2.1 (cf. Chua & Lin (1988), Theorem 3)

The string $s_0 s_1 \dots s_{N-1}$ generates an admissible periodic sequence of least period N for the nonlinear map F , with parameter value a such that $-2 < a < 2$ but $a \neq 2 \cos \frac{2k\pi}{N}$ for $k = 1, 2, \dots, N-1$, if and only if the N points x_0, x_1, \dots, x_{N-1} in \mathbb{R}^2 defined by

$$\begin{aligned} x_0 &= (I - A^N)^{-1} (s_0 A^{N-1} + s_1 A^{N-2} + \dots + s_{N-1} I) b \\ x_1 &= (I - A^N)^{-1} (s_1 A^{N-1} + s_2 A^{N-2} + \dots + s_0 I) b \\ &\vdots \\ x_{N-1} &= (I - A^N)^{-1} (s_{N-1} A^{N-1} + s_0 A^{N-2} + \dots + s_{N-2} I) b. \end{aligned} \quad (2.3)$$

all lie within the square I^2 .

In the course of the proof of Proposition 2.1 to follow, and on many occasions throughout the thesis, the analysis of the dynamical behaviour of the map F is greatly simplified by making a change of coordinates $x = Ty$ of the state-space variable x , where T is the matrix

$$T = \begin{pmatrix} 1 & 0 \\ \cos \theta & \sin \theta \end{pmatrix} \quad (2.4)$$

and θ , a new parameter for F , is related to a through $a = 2 \cos \theta$. The parameter θ assumes values in the range $0 < \theta < \pi$ to correspond with the range $-2 < a < 2$ for a . Because $Ax = T(T^{-1}AT)y$, the matrix in the new coordinate system associated with the nonlinear

map is

$$\mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} = \mathbf{R}_\theta; \quad (2.5)$$

the matrix \mathbf{R}_θ is orthogonal and represents a clockwise rotation through an angle θ in two dimensions. In this way the rotational action within the linear component \mathbf{A} of the map \mathbf{F} is isolated : because of the nature of the eigenvalues of \mathbf{A} there has to exist an appropriate coordinate system in which its matrix is a simple rotation. An immediate consequence is that the matrix $\mathbf{I} - \mathbf{A}^N$ is invertible precisely when $\theta \neq \frac{2k\pi}{N}$ for $k = 1, 2, \dots, N-1$.

When the \mathbf{x} variable is restricted to the state-space I^2 of the dynamical system, the \mathbf{y} variable takes its values in the region $\mathbf{T}^{-1}(I^2)$, which is a rhombus as indicated in Figure 2.2. The relationship $\mathbf{y} = \mathbf{T}^{-1}\mathbf{x}$ between a point $\mathbf{y} \in \mathbf{T}^{-1}(I^2)$ and its corresponding point $\mathbf{x} \in I^2$ is best expressed by writing \mathbf{T}^{-1} as the product

$$\mathbf{T}^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{\cos \theta}{\sin \theta} & \frac{1}{\sin \theta} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{\cos \theta}{\sin \theta} & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin \theta} \end{pmatrix}, \quad (2.6)$$

the combination of a vertical stretch by an amount $\frac{1}{\sin \theta}$ and a shear in the vertical direction. In the new coordinate system the nonlinear map $\mathbf{x} \mapsto \mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x} + s(\mathbf{x})\mathbf{b}$ becomes

$$\mathbf{y} \mapsto \mathbf{R}_\theta \mathbf{y} + s(\mathbf{y}) \mathbf{T}^{-1}\mathbf{b}, \quad (2.7)$$

where, of course, $s(\mathbf{y})$ denotes the application of the function (2.2) at the value $\mathbf{x} = \mathbf{T}\mathbf{y}$.

Proof of Proposition 2.1

Suppose first that all the points $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$ lie within I^2 . Then for $r = 0, 1, \dots, N-1$,

$$\begin{aligned} \mathbf{F}(\mathbf{x}_r) &= \mathbf{A}\mathbf{x}_r + s(\mathbf{x}_r)\mathbf{b} \\ &= (\mathbf{I} - \mathbf{A}^N)^{-1} (s_r \mathbf{A}^N + s_{r+1} \mathbf{A}^{N-1} + \dots + s_{r-1} \mathbf{A}) \mathbf{b} + s(\mathbf{x}_r)\mathbf{b} \\ &= (\mathbf{I} - \mathbf{A}^N)^{-1} (s_{r+1} \mathbf{A}^{N-1} + s_{r+2} \mathbf{A}^{N-2} + \dots + s_r \mathbf{I}) \mathbf{b} + (s(\mathbf{x}_r) - s_r)\mathbf{b} \\ &= \mathbf{x}_{r+1} + (s(\mathbf{x}_r) - s_r)\mathbf{b}, \end{aligned} \quad (2.8)$$

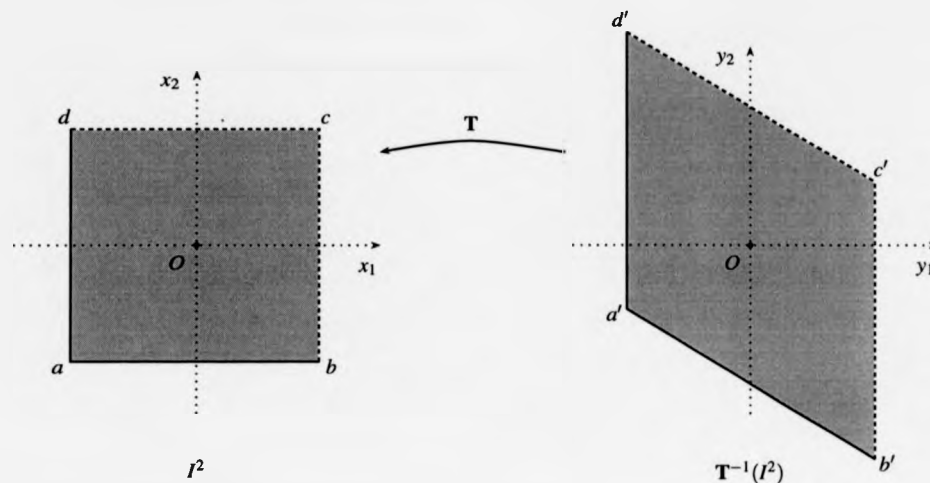


Figure 2.2: The correspondence between the domains of the nonlinear map in the two coordinate systems. Dashed lines indicate the missing boundaries, and the vertices a', b', c', d' relate to a, b, c, d via $a' = T^{-1}(a)$ etc.

where we take $\mathbf{x}_N = \mathbf{x}_0$, and it follows that $s(\mathbf{x}_r) = s_r$, because both \mathbf{x}_{r+1} and $\mathbf{F}(\mathbf{x}_r)$ lie within I^2 . Hence the point \mathbf{x}_0 , iterated under the map \mathbf{F} , gives rise to the periodic symbolic sequence generated by the string $s_0 s_1 \dots s_{N-1}$.

For the converse it is advantageous to work in the alternative coordinates introduced above. We suppose that the point \mathbf{y}_0 in $T^{-1}(I^2)$ gives rise to the periodic sequence generated by the string $s_0 s_1 \dots s_{N-1}$, and consider its orbit $\{\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \dots\}$ under the nonlinear map. For the moment we restrict our attention to the subset consisting of every N^{th} iterate and, using the periodicity of the symbolic sequence, these may be listed as

$$\mathbf{y}_0, \quad \mathbf{y}_N = \mathbf{R}_{N\theta} \mathbf{y}_0 + \mathbf{u}_0, \quad \mathbf{y}_{2N} = \mathbf{R}_{N\theta} \mathbf{y}_N + \mathbf{u}_0, \quad \mathbf{y}_{3N} = \mathbf{R}_{N\theta} \mathbf{y}_{2N} + \mathbf{u}_0, \quad \dots \quad \text{etc.},$$

where

$$\mathbf{u}_0 = (s_0 \mathbf{R}_{(N-1)\theta} + s_1 \mathbf{R}_{(N-2)\theta} + \dots + s_{N-1} \mathbf{I}) T^{-1} \mathbf{b}. \quad (2.9)$$

The iterates y_0, y_N, y_{2N}, \dots all lie on a circle with centre $\eta_0 = (\mathbf{I} - \mathbf{R}_{N\theta})^{-1} \mathbf{u}_0$. This follows because

$$\begin{aligned} y_{(k+1)N} - \eta_0 &= (\mathbf{I} - \mathbf{R}_{N\theta})^{-1} \{(\mathbf{I} - \mathbf{R}_{N\theta})y_{(k+1)N} - \mathbf{u}_0\} \\ &= (\mathbf{I} - \mathbf{R}_{N\theta})^{-1} \{(\mathbf{I} - \mathbf{R}_{N\theta})(\mathbf{R}_{N\theta}y_{kN} + \mathbf{u}_0) - \mathbf{u}_0\} \\ &= (\mathbf{I} - \mathbf{R}_{N\theta})^{-1} \mathbf{R}_{N\theta} \{(\mathbf{I} - \mathbf{R}_{N\theta})y_{kN} - \mathbf{u}_0\} \\ &= \mathbf{R}_{N\theta} (y_{kN} - \eta_0) \end{aligned} \quad (2.10)$$

for each positive integer k . We use the knowledge that the iterates lie on a circle to show that η_0 is in $\mathbf{T}^{-1}(I^2)$. There are two cases that need to be considered individually.

(i) $N\theta$ is not a rational multiple of 2π

The points y_0, y_N, y_{2N}, \dots are dense on the circle, so both the circle and its interior are contained in the closure of $\mathbf{T}^{-1}(I^2)$. Hence the centre η_0 is in $\mathbf{T}^{-1}(I^2)$.

(ii) $N\theta$ is a rational multiple of 2π

In this case $N\theta = 2\pi \frac{p}{q}$, so $q(N\theta) = 2p\pi$ and $\mathbf{R}_{N\theta}^q = (\mathbf{R}_{N\theta})^q = \mathbf{R}_{q(N\theta)}$ is the identity matrix. Now,

$$\begin{aligned} \frac{1}{q}(y_0 + y_N + \dots + y_{(q-1)N}) - \eta_0 &= \frac{1}{q}[(y_0 - \eta_0) + (y_N - \eta_0) + \dots + (y_{(q-1)N} - \eta_0)] \\ &= \frac{1}{q}[\mathbf{I} + \mathbf{R}_{N\theta} + \mathbf{R}_{N\theta}^2 + \dots + \mathbf{R}_{N\theta}^{q-1}](y_0 - \eta_0) \\ &= \frac{1}{q}(\mathbf{I} - \mathbf{R}_{N\theta})^{-1} \underbrace{(\mathbf{I} - \mathbf{R}_{N\theta}^q)}_{=0} (y_0 - \eta_0) \\ &= 0, \end{aligned}$$

so η_0 is interior to the polygon formed by joining the points $y_0, y_N, \dots, y_{(q-1)N}$. It follows by convexity that η_0 is in $\mathbf{T}^{-1}(I^2)$.

An entirely analogous argument shows that for $1 \leq r \leq N-1$, the collection of iterates $y_r, y_{N+r}, y_{2N+r}, \dots$ lie on a circle centred at $\eta_r = (\mathbf{I} - \mathbf{R}_{N\theta})^{-1} \mathbf{u}_r$ where $\mathbf{u}_r = (s_r \mathbf{R}_{(N-1)\theta} + s_{r+1} \mathbf{R}_{(N-2)\theta} + \dots + s_{r-1} \mathbf{I}) \mathbf{T}^{-1} \mathbf{b}$, and consequently that each $\eta_r \in \mathbf{T}^{-1}(I^2)$.

To complete the proof we further note that

$$\begin{aligned}
 \mathbf{T}\boldsymbol{\eta}_r &= \mathbf{T}(\mathbf{I} - \mathbf{R}_{N\theta})^{-1} \mathbf{u}_r = \mathbf{T} \left(\mathbf{T}^{-1} (\mathbf{I} - \mathbf{A}^N)^{-1} \mathbf{T} \right)^{-1} \mathbf{u}_r = (\mathbf{I} - \mathbf{A}^N)^{-1} \mathbf{T} \mathbf{u}_r \\
 &= (\mathbf{I} - \mathbf{A}^N)^{-1} (s_r \mathbf{A}^{N-1} + s_{r+1} \mathbf{A}^{N-2} + \cdots + s_{r-1} \mathbf{I}) \mathbf{b} \\
 &= \mathbf{x}_r,
 \end{aligned} \tag{2.11}$$

so the condition $\boldsymbol{\eta}_r \in \mathbf{T}^{-1}(I^2)$ corresponds to $\mathbf{x}_r \in I^2$. \square

The proof here corresponds to a more general statement of the criterion than that appearing in (Chua & Lin, 1988); our formulation additionally permits rational multiples of π for the parameter θ . The values $\theta = \frac{2k\pi}{N}$ (equivalently $a = 2\cos\frac{2k\pi}{N}$) must be excluded because in these cases the matrices $(\mathbf{I} - \mathbf{R}_{N\theta})$ and $(\mathbf{I} - \mathbf{A}^N)$ are not invertible: both reduce to the zero matrix, so we are unable to introduce the points $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$ in the manner described above. Periodic sequences that are admissible at these parameter values do exist though, and we exhibit one in the following example.

Example 2.1 : An admissible sequence of period 6 when $\theta = \frac{\pi}{3}$

Take the string $+-000-$, of length $N = 6$, at the parameter value $\theta = \frac{\pi}{3} = \frac{2\pi}{6}$. Any point $\mathbf{x}_0 = (x_1, -1)$ with $0 < x_1 < 1$ gives rise to this sequence, and is itself a period-6 point for \mathbf{F} , because

$$\begin{aligned}
 \mathbf{x}_1 &= \mathbf{F}(\mathbf{x}_0) = \mathbf{A}\mathbf{x}_0 + \mathbf{b} = (-1, -x_1 + 1) \\
 \mathbf{x}_2 &= \mathbf{F}(\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 - \mathbf{b} = (-x_1 + 1, -x_1) \\
 \mathbf{x}_3 &= \mathbf{F}(\mathbf{x}_2) = \mathbf{A}\mathbf{x}_2 = (-x_1, -1) \\
 \mathbf{x}_4 &= \mathbf{F}(\mathbf{x}_3) = \mathbf{A}\mathbf{x}_3 = (-1, x_1 - 1) \\
 \mathbf{x}_5 &= \mathbf{F}(\mathbf{x}_4) = \mathbf{A}\mathbf{x}_4 = (x_1 - 1, x_1) \\
 \mathbf{x}_6 &= \mathbf{F}(\mathbf{x}_5) = \mathbf{A}\mathbf{x}_5 - \mathbf{b} = (x_1, -1) = \mathbf{x}_0.
 \end{aligned}$$

To conclude this section we address these particular parameter values and indicate what to expect in the exceptional situation (not treated in the literature). At a parameter value $\theta = \frac{2k\pi}{N}$, $\mathbf{R}_{N\theta}$ is the identity matrix, so the iterates $\mathbf{y}_0, \mathbf{y}_N, \mathbf{y}_{2N}, \dots$ are

$$\mathbf{y}_0, \quad \mathbf{y}_N = \mathbf{y}_0 + \mathbf{u}_0, \quad \mathbf{y}_{2N} = \mathbf{y}_N + \mathbf{u}_0, \quad \mathbf{y}_{3N} = \mathbf{y}_{2N} + \mathbf{u}_0, \quad \dots$$

and for this to be consistent with the fact that all iterates are in $\mathbf{T}^{-1}(I^2)$ we require $\mathbf{u}_0 = \mathbf{0}$, i.e.

$$(s_0 \mathbf{R}_{(N-1)\theta} + s_1 \mathbf{R}_{(N-2)\theta} + \dots + s_{N-1} \mathbf{I}) \mathbf{T}^{-1} \mathbf{b} = \mathbf{0}.$$

Consideration of the iterates $\mathbf{y}_1, \mathbf{y}_{N+1}, \mathbf{y}_{2N+1}, \dots$ similarly leads to

$$(s_1 \mathbf{R}_{(N-1)\theta} + s_2 \mathbf{R}_{(N-2)\theta} + \dots + s_0 \mathbf{I}) \mathbf{T}^{-1} \mathbf{b} = \mathbf{0}$$

which can be rewritten as

$$(s_0 \mathbf{R}_{(N-1)\theta} + s_1 \mathbf{R}_{(N-2)\theta} + \dots + s_{N-1} \mathbf{I}) \mathbf{R}_\theta \mathbf{T}^{-1} \mathbf{b} = \mathbf{0}.$$

Since the vectors $\mathbf{T}^{-1} \mathbf{b}$ and $\mathbf{R}_\theta \mathbf{T}^{-1} \mathbf{b}$ are linearly independent it follows that

$$s_0 \mathbf{R}_{(N-1)\theta} + s_1 \mathbf{R}_{(N-2)\theta} + \dots + s_{N-1} \mathbf{I} = \mathbf{0} \quad (2.12)$$

with $\theta = \frac{2k\pi}{N}$.

We may easily verify that this matrix is indeed zero for the string $+-000-$ from the example: $\mathbf{R}_{\frac{5\pi}{3}} - \mathbf{R}_{\frac{4\pi}{3}} = \mathbf{I}$ follows directly from $\sin \frac{5\pi}{3} = \sin \frac{4\pi}{3}$ and $\cos \frac{5\pi}{3} - \cos \frac{4\pi}{3} = 1$. As a source of other examples we suggest those obtained by the following method to produce strings which satisfy the necessary condition (2.12). To be definite we take $N = 12$ and $\theta = \frac{\pi}{6}$; then $\mathbf{R}_{6\theta} + \mathbf{I} = \mathbf{0}$ and $\mathbf{R}_{8\theta} + \mathbf{R}_{4\theta} + \mathbf{I} = \mathbf{0}$, so that

$$s_0 \mathbf{R}_{11\theta} + s_1 \mathbf{R}_{10\theta} + \dots + s_{11} \mathbf{I} = \mathbf{0}$$

for each of the four strings $00+00++000++$, $00-00--000--$, $00+00-+000+-$ and $00-00+-000-+$. Two coprime factors of N could always be used in this way to create such strings.

Satisfying the condition (2.12) does not guarantee that a string $s_0 s_1 \dots s_{N-1}$ of length N generates an admissible periodic for $\theta = \frac{2k\pi}{N}$, and nor does it provide a direct means to locate points in state-space which give rise to the sequence, in contrast to the situation for all other values of θ where the point \mathbf{x}_0 arose naturally out of the method. However we can at least learn where to look for such points, for one consequence of the conditions $\mathbf{R}_{N\theta} = \mathbf{I}$ and $s_0 \mathbf{R}_{(N-1)\theta} + s_1 \mathbf{R}_{(N-2)\theta} + \dots + s_{N-1} \mathbf{I} = \mathbf{0}$ is that *any* point which gives rise to a sequence commencing $s_0 s_1 \dots s_{N-1}$ is a periodic point of period N . This follows because the N^{th} iterate of such a point under the nonlinear map is $\mathbf{R}_{N\theta} \mathbf{y} + \mathbf{u}_0 = \mathbf{y}$, and permits a search for these points using the techniques we describe in Chapter 3.

2.1.3 Orbits of points associated with admissible periodic sequences

A feature of the proof of Proposition 2.1 was a full description of the orbit of any point which gives rise to some prescribed admissible periodic sequence. More specifically, working in the \mathbf{y} -coordinate plane, we proved the following about the orbit of a point \mathbf{y}_0 defining the periodic sequence generated by the string $s_0 s_1 \dots s_{N-1}$:

- (i) The iterates of \mathbf{y}_0 lie on N congruent circles of radius $r = \|\mathbf{y}_0 - \boldsymbol{\eta}_0\|$ whose centres are at the points $\boldsymbol{\eta}_0, \boldsymbol{\eta}_1, \dots, \boldsymbol{\eta}_{N-1}$. The locations of the centres depend only on the string $s_0 s_1 \dots s_{N-1}$ and the value of the parameter, and are determined by

$$\boldsymbol{\eta}_r = (\mathbf{I} - \mathbf{R}_{N\theta})^{-1} \mathbf{u}_r$$

where

$$\mathbf{u}_r = (s_r \mathbf{R}_{(N-1)\theta} + s_{r+1} \mathbf{R}_{(N-2)\theta} + \dots + s_{N-1} \mathbf{I}) \mathbf{T}^{-1} \mathbf{b}.$$

- (ii) The points $\mathbf{y}_r, \mathbf{y}_{N+r}, \mathbf{y}_{2N+r}, \dots$ lie on the r^{th} circle (centred at $\boldsymbol{\eta}_r$), each is obtained from its predecessor here by a rotation around the circle through an angle $N\theta$. When θ is not a rational multiple of π we obtain a dense set, otherwise a finite number of equally spaced points.
- (iii) Successively the iterates $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2, \dots$ lie one by one on the N circles visited in turn.

A corresponding description of the orbits in the original state-space coordinates may be obtained by applying the transformation $\mathbf{x} = \mathbf{T}\mathbf{y}$, and is of course the same but for this change in geometry. The contracting and shearing effect of \mathbf{T} on the vertical coordinate sends each circle to an ellipse inclined at $\frac{\pi}{4}$ to the horizontal as we now demonstrate.

To describe the ellipses obtained, we note that in the standard coordinate geometry of the xy -plane the equation of an ellipse having the x -axis as its major axis and the y -axis as its minor axis is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where a and b are the lengths of the semi-major and semi-minor axes. Thus when this ellipse is rotated anticlockwise by $\frac{\pi}{4}$, so that the line $y = x$ is now the major axis, its equation is instead

$$\frac{1}{2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) x^2 + \left(\frac{1}{a^2} - \frac{1}{b^2} \right) xy + \frac{1}{2} \left(\frac{1}{a^2} + \frac{1}{b^2} \right) y^2 = 1. \quad (2.13)$$

The circle in the y -coordinate plane with centre $\boldsymbol{\eta}_r = (\eta_{r,1}, \eta_{r,2})$ and radius δ is

$$(y_1 - \eta_{r,1})^2 + (y_2 - \eta_{r,2})^2 = \delta^2,$$

i.e. $\|\mathbf{y} - \boldsymbol{\eta}_r\|^2 = \delta^2.$

The curve in state-space coordinates that corresponds to this via the transformation \mathbf{T} is

$$\begin{aligned} \|\mathbf{T}^{-1}\mathbf{x} - \boldsymbol{\eta}_r\|^2 &= \delta^2, \\ \text{i.e. } \|\mathbf{T}^{-1}(\mathbf{x} - \mathbf{x}_r)\|^2 &= \delta^2 \quad \text{where } \mathbf{T}\boldsymbol{\eta}_r = \mathbf{x}_r, \\ \text{i.e. } (\mathbf{x} - \mathbf{x}_r)^T (\mathbf{T}^{-1})^T \mathbf{T}^{-1} (\mathbf{x} - \mathbf{x}_r) &= \delta^2, \\ \text{i.e. } \frac{1}{\sin^2 \theta} (x_1 - x_{r,1})^2 - 2 \frac{\cos \theta}{\sin^2 \theta} (x_1 - x_{r,1})(x_2 - x_{r,2}) + \frac{1}{\sin^2 \theta} (x_2 - x_{r,2})^2 &= \delta^2, \end{aligned} \quad (2.14)$$

so from (2.13) it is an ellipse, centre at \mathbf{x}_r with

$$\text{semi-major axis} = \delta \sqrt{1 + \cos \theta} = \delta \sqrt{1 + \frac{a}{2}}, \quad (2.15a)$$

$$\text{semi-minor axis} = \delta \sqrt{1 - \cos \theta} = \delta \sqrt{1 - \frac{a}{2}}. \quad (2.15b)$$

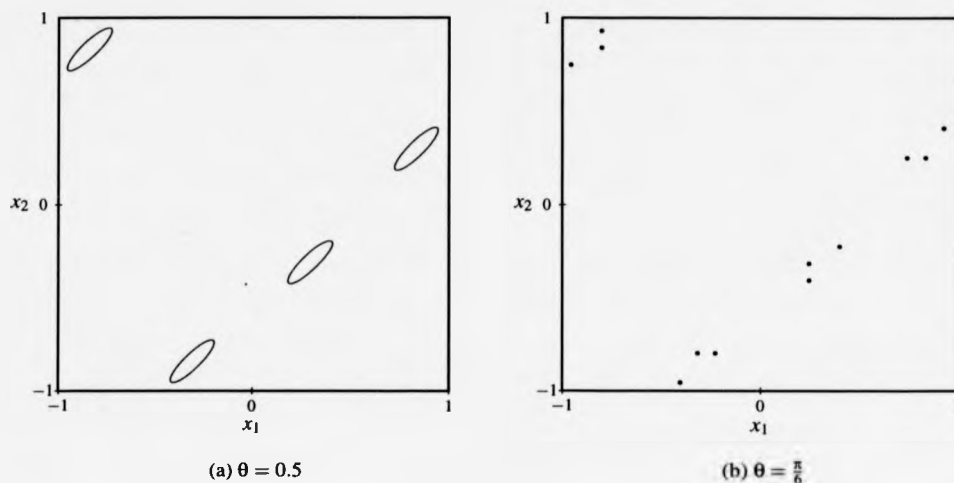


Figure 2.3: The point $(-0.4, -0.95)$ gives rise to the periodic sequence generated by $+-00$ for both parameter values. The plots of the orbits have a different character, the distinction depending on whether θ is a rational multiple of π .

Figures 2.3(a) and 2.3(b) are plots of sample orbits in the state-space originating at a point giving rise to a periodic sequence; for both plots the periodic sequence is $+-00$. Figure 2.3(a) shows the typical situation where θ is not a rational multiple of π ($\theta = 0.5$, $a = 2\cos 0.5$), in this example the orbit lies dense on four ellipses. The alternative situation, where θ is a rational multiple of π , is illustrated in Figure 2.3(b) ($\theta = \frac{\pi}{6}$, $a = \sqrt{3}$), and here the iterates periodically visit 12 isolated points.

In addition to describing the orbits, once a periodic sequence has been shown to be admissible we can specify precisely the region of points in I^2 which give rise to that sequence. We start with the case where θ is not a rational multiple of π : for these values Proposition 2.2 below shows that the regions are elliptical in form. To state the result concisely we introduce some notation for the elliptical regions: let $E(x_r, \delta)$ denote the solid ellipse whose boundary is given by the formula (2.14). Due to its orientation, this ellipse touches the four

sides of the square whose centre is \mathbf{x}_r , whose sides are parallel to the coordinate axes and whose side length is 2δ . Hence if ρ is the minimum distance of any of the points $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$ from a boundary of I^2 then the corresponding solid ellipses $E(\mathbf{x}_r, \rho)$ are contained in the closure of I^2 and are collectively maximal subject to this property.

Proposition 2.2 (cf. Chua and Lin 1988, Theorem 5)

For a parameter value $a = 2\cos\theta$ such that θ is not a rational multiple of π , the region of points in state-space which give rise to the admissible periodic sequence generated by a given string $s_0s_1\dots s_{N-1}$ consists of the interior of the solid ellipse $E(\mathbf{x}_0, \rho)$ together with some points of the boundary. Two possibilities arise: either the entire boundary is included, or some dense countable subset of the boundary must be excluded.

Proof

We have already shown that if \mathbf{x} gives rise to the periodic sequence generated by the string $s_0s_1\dots s_{N-1}$ then \mathbf{x} belongs to the solid ellipse $E(\mathbf{x}_0, \rho)$. The rest of the proof splits into two parts: the first (a) is to show that if \mathbf{x} belongs to the interior of $E(\mathbf{x}_0, \rho)$ then \mathbf{x} gives rise to the periodic sequence generated by $s_0s_1\dots s_{N-1}$, and the second (b) treats the case of the boundary points. We work in the transformed coordinates introduced in Section 2.1.2.

- (a) Suppose \mathbf{y} is the point in $\mathbf{T}^{-1}(I^2)$ corresponding to our \mathbf{x} within the interior of $E(\mathbf{x}_0, \rho)$; then $\|\mathbf{y} - \boldsymbol{\eta}_0\| < \rho$. Under the nonlinear map $\boldsymbol{\eta}_0$ is sent to $\boldsymbol{\eta}_1$. Consider the isometry $\mathbf{z} \mapsto \mathbf{R}_\theta \mathbf{z} + s_0 \mathbf{T}^{-1} \mathbf{b}$; under this isometry $\boldsymbol{\eta}_0$ maps to $\boldsymbol{\eta}_1$ (because for the point $\boldsymbol{\eta}_0$ the isometry has the same effect as the nonlinear map). Suppose $\mathbf{y} \mapsto \mathbf{y}'$ under the isometry, then \mathbf{y}' is interior to the circle centre $\boldsymbol{\eta}_1$ and radius ρ , in particular $\mathbf{y}' \in \mathbf{T}^{-1}(I^2)$. Consequently the nonlinear map applied to \mathbf{y} gives \mathbf{y}' , because there is only one choice for s so that $\mathbf{R}_\theta \mathbf{y} + s \mathbf{T}^{-1} \mathbf{b}$ belongs to $\mathbf{T}^{-1}(I^2)$. Hence the first digit of the symbolic sequence corresponding to \mathbf{y} is s_0 . Subsequent digits are confirmed in the same fashion.
- (b) From the definition of ρ , at least one of the circles $\|\mathbf{y} - \boldsymbol{\eta}_r\| = \rho$ meets the boundary of $\mathbf{T}^{-1}(I^2)$. It could happen that all these circles nevertheless are within $\mathbf{T}^{-1}(I^2)$ in which case the argument in (a) shows that all the points \mathbf{y} such that $\|\mathbf{y} - \boldsymbol{\eta}_0\| = \rho$ give rise to

the periodic sequence generated by the string $s_0 s_1 \dots s_{N-1}$. This means that the entire boundary of $E(x_0, \rho)$ consists of points which give rise to this sequence.

Otherwise at least one of the circles has points (either 1 or 2 in number) which do not belong to the set $T^{-1}(I^2)$. Should the circle centred at η_0 have such points then, certainly, these are excluded because they do not belong to the domain of the nonlinear map. If next the circle centred at η_1 has such points, then the points on the circle centred at η_0 that are the inverse images of these under the affine map considered in (a) do not produce the digit s_0 when the nonlinear map is applied, and therefore do not give rise to the periodic sequence generated by the string $s_0 s_1 \dots s_{N-1}$. Continue in this way : look at all the inverse images (by the affine map that uses s_1 in place of s_0), on the circle centred at η_1 , which derive from the points on the circle centred at η_2 not belonging to $T^{-1}(I^2)$. Their inverse images in turn by the first affine map are points on the circle centred at η_0 which under two iterations of the nonlinear map do not produce the digit pair $s_0 s_1$. Extend this scheme so that all the circles are visited and by this stage we have excluded all the points on the circle centred at η_0 which do not give rise to the string of initial digits $s_0 s_1 \dots s_{N-1}$ under the nonlinear map. Next we must exclude those points which do not give rise to $s_0 s_1 \dots s_{N-1} s_0$, and for this we need to work back from the circle centred at η_0 itself, through all the other circles until we arrive again at the circle centred at η_0 . If we continue indefinitely the process of taking these inverse images, then we obtain a countable subset of points on the circle centred at η_0 , precisely those which do not give rise under iteration of the nonlinear map to the periodic sequence generated by the string $s_0 s_1 \dots s_{N-1}$.

The collection of points arising from the construction described is dense on the circle centred at η_0 . The reason is that if y is one of these points then so is y' , the point obtained by rotating y around the circle through an angle $-N\theta$. For otherwise we know from the proof of Proposition 2.1 that the N^{th} iterate of y' under the nonlinear map brings us back to y , and so we obtain a contradiction because y does not give rise to the periodic sequence generated $s_0 s_1 \dots s_{N-1}$. \square

In the case where θ is a rational multiple of π , the region of points which give rise to an admissible periodic sequence is entirely different in character and the corresponding result is included in our next proposition.

Proposition 2.3

For a parameter value $a = 2 \cos \theta$ where $\theta = 2\pi \frac{p}{q}$ (and excluding the exceptional values $\frac{2k\pi}{N}$ treated in Section 2.1.2), the region of points in state-space which give rise to the admissible periodic sequence generated by a given string $s_0 s_1 \dots s_{N-1}$ is a convex polygonal set containing \mathbf{x}_0 . The set is determined by at most l linear inequalities, some of which may be strict, and the value $l = \text{lcm}(q, N)$.

Proof

We know from Proposition 2.1 that \mathbf{x}_0 is a periodic point which gives rise to the periodic sequence generated by the string $s_0 s_1 \dots s_{N-1}$. In the reasoning that follows it will be convenient to extend notation and define $s_r = s_{r \bmod N}$ when $r \geq N$. If we apply \mathbf{F} to \mathbf{x}_0 l times then because of periodicity

$$\mathbf{x}_0 = \mathbf{A}^l \mathbf{x}_0 + (s_0 \mathbf{A}^{l-1} + s_1 \mathbf{A}^{l-2} + \dots + s_{l-1} \mathbf{I}) \mathbf{b}$$

but $\mathbf{A}^l = \mathbf{I}$ and therefore

$$(s_0 \mathbf{A}^{l-1} + s_1 \mathbf{A}^{l-2} + \dots + s_{l-1} \mathbf{I}) \mathbf{b} = \mathbf{0}. \quad (2.16)$$

Let \mathbf{x} be any point in I^2 which under iteration by \mathbf{F} gives rise to the periodic sequence generated by the string $s_0 s_1 \dots s_{N-1}$. Then

$$\begin{aligned} \mathbf{F}^l(\mathbf{x}) &= \mathbf{A}^l \mathbf{x} + (s_0 \mathbf{A}^{l-1} + s_1 \mathbf{A}^{l-2} + \dots + s_{l-1} \mathbf{I}) \mathbf{b} \\ &= \mathbf{x}, \end{aligned} \quad (2.17)$$

and in particular \mathbf{x} is periodic. Note also that \mathbf{x} satisfies the linear inequalities implied by the statements

$$\begin{aligned}
 \mathbf{x} &\in I^2, \\
 \mathbf{Ax} + s_0\mathbf{b} &\in I^2, \\
 \mathbf{A}^2\mathbf{x} + (s_0\mathbf{A} + s_1\mathbf{I})\mathbf{b} &\in I^2, \\
 &\vdots \\
 \mathbf{A}^{l-1}\mathbf{x} + (s_0\mathbf{A}^{l-1} + s_1\mathbf{A}^{l-2} + \cdots + s_{l-1}\mathbf{I})\mathbf{b} &\in I^2,
 \end{aligned} \tag{2.18}$$

(each such coordinate inequality will be of the form $-1 \leq \lambda x_1 + \mu x_2 + \nu < 1$).

Conversely if we look at the iterates under \mathbf{F} of a point \mathbf{x} satisfying (2.18), the first l digits of its symbolic sequence are $s_0 s_1 \dots s_{l-1}$,

$$\mathbf{F}^l(\mathbf{x}) = \mathbf{A}^l\mathbf{x} + (s_0\mathbf{A}^{l-1} + s_1\mathbf{A}^{l-2} + \cdots + s_{l-1}\mathbf{I})\mathbf{b} = \mathbf{x},$$

and so the symbolic sequence continues periodically. □

Propositions 2.2 and 2.3 allow us to describe the regions for the examples we studied earlier in this section involving the period-4 sequence generated by $+-00$. When $\theta = 0.5$ ($a = 2\cos 0.5$), we obtain the ellipse $E(\mathbf{x}_0, \rho)$ centred on $\mathbf{x}_0 = \left(\frac{-2}{a^2+2a}, \frac{-2a-2}{a^2+2a}\right)$ and whose size is determined from $\rho = \frac{a^2-2}{a^2+2a}$. The solid ellipse appears as the shaded region in Figure 2.4(a). In the other case, when $\theta = \frac{\pi}{6}$, $l = \text{lcm}(12, 4) = 12$ and the inequalities are :

- | | |
|--|--|
| 1. $-1 \leq x_1 < 1$ | 7. $-1 \leq -x_1 + 2(2 - \sqrt{3}) < 1$ |
| 2. $-1 \leq x_2 < 1$ | 8. $-1 \leq -x_2 + 2(\sqrt{3} - 2) < 1$ |
| 3. $-1 \leq -x_1 + \sqrt{3}x_2 + 2 < 1$ | 9. $-1 \leq x_1 - \sqrt{3}x_2 + 2(1 - \sqrt{3}) < 1$ |
| 4. $-1 \leq -\sqrt{3}x_1 + 2x_2 + 2(\sqrt{3} - 1) < 1$ | 10. $-1 \leq \sqrt{3}x_1 - 2x_2 - 2 < 1$ |
| 5. $-1 \leq -2x_1 + \sqrt{3}x_2 + 2(2 - \sqrt{3}) < 1$ | 11. $-1 \leq 2x_1 - \sqrt{3}x_2 < 1$ |
| 6. $-1 \leq -\sqrt{3}x_1 + x_2 + 2(\sqrt{3} - 2) < 1$ | 12. $-1 \leq \sqrt{3}x_1 - x_2 < 1$ |

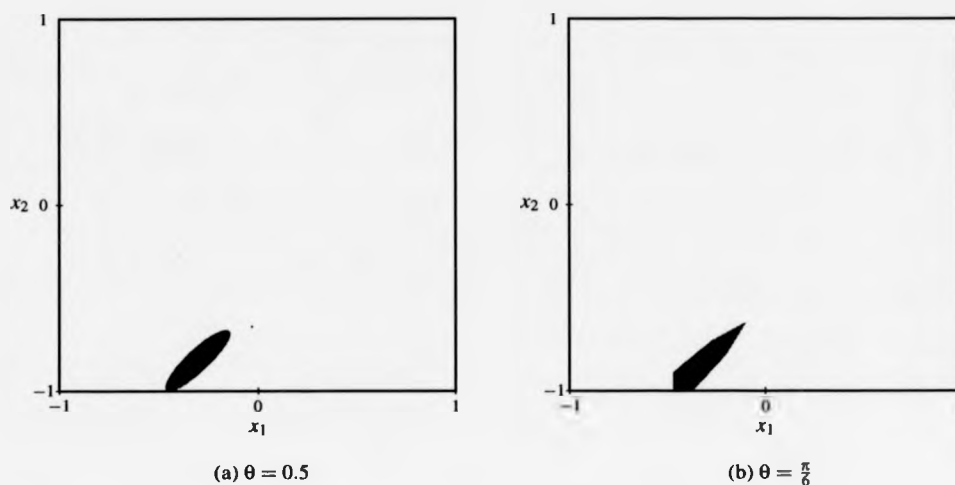


Figure 2.4: The regions in state-space consisting of the points that give rise to the sequence $+-00$. The typical case (θ is not a rational multiple of π) is a solid ellipse, as in (a); the alternative is a convex polygonal set, as in (b).

Only 6 of these are instrumental in delimiting the region, namely the left inequalities of 2, 6, 10 and the right inequalities of 3, 7, 11. The polygonal region enclosed is shown in Figure 2.4(b).

2.1.4 The trigonometric inequalities of Chua and Lin : a criterion for admissibility

We have seen in Proposition 2.1 that a necessary and sufficient condition for a string $s_0 s_1 \dots s_{N-1}$ to generate an admissible periodic sequence is that each of the points $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$ defined there lie within the square I^2 (provided of course a is not one of the pathological values $2 \cos \frac{2k\pi}{N}$ discussed at the end of Section 2.1.2, henceforth never considered). For numerical work where one might need to check a particular string at a particular parameter value, the foregoing is adequate. However for more sophisticated tasks a reformulation

of this condition, due also to Chua and Lin, as a set of N trigonometric inequalities, turns out to be very effective.

For the deduction of these inequalities we start with the formula (2.3) for \mathbf{x}_0 ,

$$\mathbf{x}_0 = (\mathbf{I} - \mathbf{A}^N)^{-1} (s_0 \mathbf{A}^{N-1} + s_1 \mathbf{A}^{N-2} + \cdots + s_{N-1} \mathbf{I}) \mathbf{b}$$

and rewrite it in terms of the parameter θ , where $a = 2 \cos \theta$. First we calculate the matrix $(\mathbf{I} - \mathbf{A}^N)^{-1} (s_0 \mathbf{A}^{N-1} + s_1 \mathbf{A}^{N-2} + \cdots + s_{N-1} \mathbf{I})$, conjugated by \mathbf{T} , i.e.

$$\begin{aligned} \mathbf{T}^{-1} (\mathbf{I} - \mathbf{A}^N)^{-1} (s_0 \mathbf{A}^{N-1} + s_1 \mathbf{A}^{N-2} + \cdots + s_{N-1} \mathbf{I}) \mathbf{T} \\ = (\mathbf{I} - \mathbf{R}_{N\theta})^{-1} (s_0 \mathbf{R}_{(N-1)\theta} + s_1 \mathbf{R}_{(N-2)\theta} + \cdots + s_{N-1} \mathbf{I}). \end{aligned} \quad (2.19)$$

Now,

$$\mathbf{I} - \mathbf{R}_{N\theta} = \begin{pmatrix} 1 - \cos N\theta & -\sin N\theta \\ \sin N\theta & 1 - \cos N\theta \end{pmatrix} = 2 \sin \frac{N}{2} \theta \begin{pmatrix} \sin \frac{N}{2} \theta & -\cos \frac{N}{2} \theta \\ \cos \frac{N}{2} \theta & \sin \frac{N}{2} \theta \end{pmatrix} \quad (2.20)$$

and its inverse is

$$(\mathbf{I} - \mathbf{R}_{N\theta})^{-1} = \frac{1}{2 \sin \frac{N}{2} \theta} \begin{pmatrix} \sin \frac{N}{2} \theta & \cos \frac{N}{2} \theta \\ -\cos \frac{N}{2} \theta & \sin \frac{N}{2} \theta \end{pmatrix} = \frac{1}{2 \sin \frac{N}{2} \theta} \mathbf{R}_{-\frac{N}{2}\theta} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \quad (2.21)$$

Resuming now with \mathbf{x}_0 , and using $\mathbf{T} = \begin{pmatrix} 1 & 0 \\ \cos \theta & \sin \theta \end{pmatrix}$, $\mathbf{T}^{-1} \mathbf{b} = \begin{pmatrix} 0 \\ \frac{2}{\sin \theta} \end{pmatrix}$, we can express \mathbf{x}_0 as

$$\begin{aligned} \mathbf{x}_0 &= \mathbf{T} \frac{1}{2 \sin \frac{N}{2} \theta} (s_0 \mathbf{R}_{(\frac{N}{2}-1)\theta} + s_1 \mathbf{R}_{(\frac{N}{2}-2)\theta} + \cdots + s_{N-1} \mathbf{R}_{-\frac{N}{2}\theta}) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ \frac{2}{\sin \theta} \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ \cos \theta & \sin \theta \end{pmatrix} \frac{1}{\sin \frac{N}{2} \theta \sin \theta} (s_0 \mathbf{R}_{(\frac{N}{2}-1)\theta} + s_1 \mathbf{R}_{(\frac{N}{2}-2)\theta} + \cdots + s_{N-1} \mathbf{R}_{-\frac{N}{2}\theta}) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \end{aligned}$$

In particular its first coordinate is given by the formula

$$\frac{1}{\sin \frac{N}{2} \theta \sin \theta} (s_0 \cos (\frac{N}{2} - 1) \theta + s_1 \cos (\frac{N}{2} - 2) \theta + \cdots + s_{N-1} \cos (-\frac{N}{2}) \theta). \quad (2.22)$$

The same rearrangement may be effected for the expressions for the other $(N - 1)$ points $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{N-1}$, and the resulting formula for the first coordinate of a general \mathbf{x}_r is obtained from (2.22) simply by cycling $s_0 \rightarrow s_1 \rightarrow s_2 \rightarrow \dots \rightarrow s_{N-1} \rightarrow s_0$, and is

$$\frac{1}{\sin \frac{N}{2} \theta \sin \theta} (s_r \cos (\frac{N}{2} - 1) \theta + s_{r+1} \cos (\frac{N}{2} - 2) \theta + \dots + s_{r-1} \cos (-\frac{N}{2}) \theta). \quad (2.23)$$

It follows directly from the definition of the nonlinear map (i.e. from $\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x} + s(\mathbf{x})\mathbf{b}$ where $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$) that the first coordinate of the point \mathbf{x}_{r+1} is the same as the second coordinate of \mathbf{x}_r . Hence the condition from Proposition 2.1, namely that $\mathbf{x}_r \in I^2$ for $r = 0, 1, \dots, N - 1$, is equivalent to the inequalities

$$-1 \leq \frac{s_r \cos (\frac{N}{2} - 1) \theta + s_{r+1} \cos (\frac{N}{2} - 2) \theta + \dots + s_{r-1} \cos (-\frac{N}{2}) \theta}{\sin \frac{N}{2} \theta \sin \theta} < 1 \quad (2.24)$$

for $r = 0, 1, \dots, N - 1$, and consequently a string $s_0 s_1 \dots s_{N-1}$ generates an admissible periodic sequence, for the parameter $a = 2 \cos \theta$, if and only if it satisfies these N inequalities.

Despite their relatively simple appearance, it should be remembered that implicit in these inequalities is the complex partitioning of that portion of the state-space of the original dynamical system that is attributable to the orbits of the points which give rise to periodic sequences. (See for example Figure 2.1 in which each elliptical island shown there consists of the points associated with some individual periodic sequence.) Apart from their original appearance in the literature (Chua & Lin, 1990b), these inequalities seem subsequently not to have been followed up. Certainly they are subtle and do not easily reveal their content; however we shall see that they form the starting point for several lines of investigation, each of which has resulted in an enhanced understanding of some aspect of admissible periodic sequences.

Whereas earlier work has focused on determining which periodic sequences are admissible at a single parameter value, our primary concern is to determine whole intervals of parameter values throughout which the admissibility of a periodic sequence persists. With the hindsight of our results we can see that the former approach does not come near to elucidating any structure present amongst admissible periodic sequences. It would appear that

only by relating the structure of the periodic sequences to the ranges of parameter values for which they are admissible does one begin to get some insight into the character of the collection of admissible periodic sequences.

A first consequence of the inequalities (2.24) is that we can give a qualitative description of the set of values of θ with which we are concerned (i.e. where a string $s_0 s_1 \dots s_{N-1}$ generates an admissible periodic sequence). To simplify the ensuing discussion, and for our later work, we introduce some notation for combinations of the trigonometric terms appearing in the inequalities. We define

$$f_0(\theta) = s_0 \cos \frac{N}{2} \theta + s_1 \cos \left(\frac{N}{2} - 1 \right) \theta + \dots + s_{N-1} \cos \left(\frac{N}{2} - 1 \right) \theta \quad (2.25)$$

and more generally

$$f_r(\theta) = \sum_{i=0}^{N-1} s_{r+i} \cos \frac{N-2i}{2} \theta, \quad (2.26)$$

for $r = 0, 1, \dots, N-1$, where as previously $s_i = s_{i \bmod N}$ when $i \geq N$. Likewise let

$$g(\theta) = \sin \frac{N}{2} \theta \sin \theta. \quad (2.27)$$

In these terms the inequalities (2.24) may be written

$$-1 \leq \frac{f_r(\theta)}{g(\theta)} < 1 \quad \text{for } r = 0, 1, \dots, N-1. \quad (2.28)$$

(Note that an inconsequential feature of our notation is that some cycling has occurred and the inequality which originally came first is now the final one.)

The trigonometric identity $\cos mA = 2 \cos(m-1)A \cos A - \cos(m-2)A$ implies that $\cos mA$ is a polynomial in $\cos A$ of degree m , so that

$$f_0(\theta) = s_0 \cos N \frac{1}{2} \theta + s_1 \cos(N-2) \frac{1}{2} \theta + \dots + s_{N-1} \cos(N-2) \frac{1}{2} \theta$$

is a polynomial of degree N in $\cos \frac{1}{2} \theta$, as are all the f_r 's, and

$$g(\theta) = \sin \frac{N}{2} \theta \sin \theta = \frac{1}{2} (\cos(N-2) \frac{1}{2} \theta - \cos(N+2) \frac{1}{2} \theta)$$

is a polynomial of degree $(N + 2)$ in $\cos \frac{1}{2}\theta$. Thus each left-hand inequality $-1 \leq \frac{f_r(\theta)}{g(\theta)}$ is equality for a finite collection of θ values only, and the set of θ values where all the inequalities hold strictly is a finite union of open intervals. We will be concerned with these open intervals of θ values within which all the inequalities hold strictly. Any value of θ where all the inequalities hold, but some left-hand ones with equality, will either be an end-point of one of these open intervals or, exceptionally, an isolated point (i.e. to either side some inequality fails). For a given string $s_0 s_1 \dots s_{N-1}$ and one of these latter θ values, the point set in state-space associated with the admissible periodic sequence generated by the string degenerates to a line segment of periodic points, or a single such point, on the left-hand or lower boundary of I^2 . These points account only minimally for the entire collection of orbits present in the state-space, whereas for a θ value from one of the open intervals the point set is a non-degenerate ellipse or polygon, characteristic of what is typically observed in computer plots for this dynamical system. Since our aim is to relate strings to intervals of admissibility for the sequences they generate, we can clearly restrict attention solely to open intervals.

It is relevant to comment here that our point of view parallels those adopted by other authors; Kocarev *et al.* (1996) take care to exclude, in the statements of their theorems, all orbits that do not remain within the interior of I^2 ; Ashwin (1997) excises from the state-space all pre- and post-images of boundary points. To clarify the discussion we give an example of the type of situation that is excluded.

Example 2.2 : A periodic sequence admissible for an isolated value of θ

For the string -0 , $g(\theta) = \sin^2 \theta$, $f_0(\theta) = -\cos \theta$ and $f_1(\theta) = -1$. Then

$$\begin{aligned} -1 \leq \frac{f_0(\theta)}{g(\theta)} < 1 & \text{ is satisfied for } \frac{\pi}{4} \leq \theta < \frac{3\pi}{4} \\ -1 \leq \frac{f_1(\theta)}{g(\theta)} < 1 & \text{ is satisfied for } \theta = \frac{\pi}{2} \text{ alone.} \end{aligned}$$

The points that give rise to the period-2 sequence generated by -0 are the left-hand bound-

any points $\mathbf{x}_0 = (-1, x_2)$ with $-1 < x_2 < 1$, because of Proposition 2.3 and

$$\mathbf{x}_1 = \mathbf{F}(\mathbf{x}_0) = \mathbf{A}\mathbf{x}_0 - \mathbf{b} = (x_2, -1),$$

$$\mathbf{x}_2 = \mathbf{F}(\mathbf{x}_1) = \mathbf{A}\mathbf{x}_1 = (-1, -x_2),$$

$$\mathbf{x}_3 = \mathbf{F}(\mathbf{x}_2) = \mathbf{A}\mathbf{x}_2 - \mathbf{b} = (-x_2, -1),$$

$$\mathbf{x}_4 = \mathbf{F}(\mathbf{x}_3) = \mathbf{A}\mathbf{x}_3 = (-1, x_2) = \mathbf{x}_0.$$

For our subsequent work we summarise in a convenient form below the trigonometric inequalities developed in this section. Because of the reasons given above we have opted for *strict inequalities*, so that the phrase “admissible periodic sequence” now applies in a slightly more restricted manner than previously but, as we have explained, we are never concerned with parameter values where there can be any difference between the two usages. For ease of reference we will call these remarkable inequalities the “Chua-Lin inequalities”.

The string $s_0 s_1 \dots s_{N-1}$ generates an admissible periodic sequence at parameter value θ if and only if

$$-|g(\theta)| < f_r(\theta) < |g(\theta)| \quad \text{for } r = 0, 1, \dots, N-1$$

$$\text{where } f_0(\theta) = s_0 \cos \frac{N}{2} \theta + s_1 \cos \left(\frac{N}{2} - 1 \right) \theta + \dots + s_{N-1} \cos \left(\frac{N}{2} - 1 \right) \theta,$$

$$f_r(\theta) = \sum_{i=0}^{N-1} s_{r+i} \cos \frac{N-2i}{2} \theta,$$

$$g(\theta) = \sin \frac{N}{2} \theta \sin \theta$$

The Chua-Lin inequalities as a criterion for admissibility

2.2 Origins for the investigation of admissible periodic sequences

2.2.1 Nonlinear dynamics arising out of overflow oscillations in digital filters

The subject of our study originated in the late 1960's, from the observation of unexpected oscillatory behaviour in the output signals of certain digital filters. When the input signal is removed, the output does not always decay to zero, as is predicted by the linear theory used quite generally to model digital filter operation. Instead the output of the filter sometimes enters a regime of self-sustained large amplitude oscillations. The systems concerned were second order digital filters, realised in direct form. These are essentially the simplest possible filters to be of use in signal processing, but are nevertheless important since they form a building block for many more complex filter designs. Figure 2.5 shows, diagrammatically, the composition of a second order direct form filter, synthesized from delay units, signal multipliers and an addition unit. In accordance with the linear theory, the output (y_n) of the filter relates to the present input (x_n), and also the inputs and outputs for the two previous time steps, via the relation

$$y_n = i_0x_n + i_1x_{n-1} + i_2x_{n-2} + ay_{n-1} + by_{n-2}, \quad (2.29)$$

where i_0, i_1, i_2, a, b are the multiplier values. Since our interest is in the behaviour of the output after an input signal has been removed, the input section of the filter plays no role, and consequently we can reformulate the relation more simply as

$$y_n = ay_{n-1} + by_{n-2}, \quad (2.30)$$

provided we initialise this recurrence by supplying two starting values, for y_{-1} and y_{-2} , so as to simulate the final internal state of the filter prior to termination of the effect of the input.

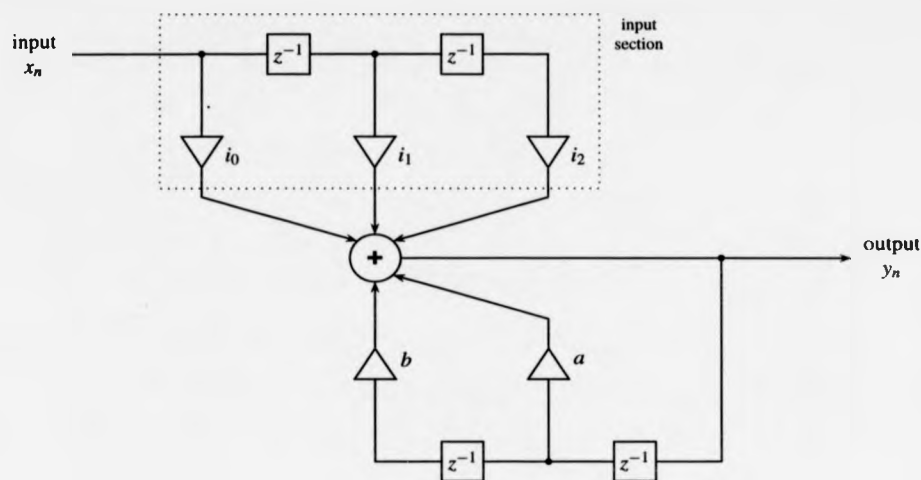


Figure 2.5: Schematic representation of a direct form second order digital filter.

The oscillations are an artifact of the physical realisation of the digital filter (this realisation may be via dedicated hardware components, or as a computer program), and occur because of nonlinearity introduced through loss of arithmetic precision when working with some, necessarily limited, scheme of machine represented numbers. In practice the operation of a digital filter departs from an idealised linear model in the following respects :

- (a) **Overflow** : There is a greatest magnitude M for the numbers that can be represented. Values generated in the multipliers, or in the addition unit, that exceed M must be handled by some overflow convention. Frequently the arithmetic system would be 2's-complement arithmetic, and within this scheme the represented numbers are integers ranging from $-M$ to $M - 1$. Overflow handling is based on the rule that 1 added to the largest positive value $M - 1$ yields $-M$.
- (b) **Rounding** : Arithmetic operations may not be performed exactly, with a potential loss of precision at each multiplication or addition step. Typically, this is commonplace with a fixed point or a floating point arithmetic system.
- (c) **Representational Error** : Because of the coarseness of the machine supported

arithmetic, it may not be possible to employ the precise multiplier values derived from the filter performance specification. (The operation of the filter may be sensitively dependent on these values.)

In the presence of any of these, the digital filter can no longer be modelled as a linear system.

The concern in this thesis is with the effect of the overflow nonlinearity introduced by 2's complement arithmetic; it is this that is responsible for the oscillations which were originally observed when the phenomenon was uncovered. A convenient way to isolate and *analyse* this particular cause of nonlinearity is to model the underlying number system through the vehicle of real numbers between -1 and 1 with arithmetic performed modulo 2: in effect the greatest magnitude for a number M is normalised to 1. In this idealisation the quantities y_n are real-valued and restricted so that $-1 \leq y_n < 1$. In place of the original linear relation (2.30), the operation of a second order direct-form filter is now described by

$$y_n = f(ay_{n-1} + by_{n-2}), \quad (2.31)$$

where, implementing the effect of the 2's complement overflow, f is the function

$$f(x) = ((x + 1) \bmod 2) - 1. \quad (2.32)$$

The graph of f has sawtooth form, as shown in Figure 2.6. One of the first investigations into the behaviour of this system, bringing to light the differences from the associated linear model, is Ebert *et al.* (1969), in which some simple overflow oscillations are exhibited.

The major preoccupation of the early studies was the elimination of overflow oscillations, which are regarded as highly undesirable in digital filter applications. Ebert *et al.* (1969) show that the filter output coincides with that for the linear model provided the multiplier coefficients a, b are such that $|a| + |b| < 1$. However this condition imposes too severe a limitation: to cover the full range of frequency response characteristics achievable by a second order filter, a and b must be permitted to assume any values in the triangular region determined by $b > -1$, $b < a + 1$, and $b < -a + 1$.

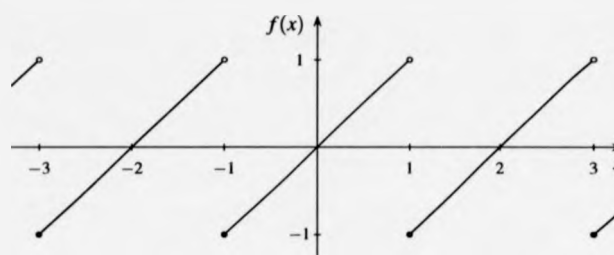


Figure 2.6: The function $f(x) = ((x + 1) \bmod 2) - 1$, used to model the effect of 2's complement arithmetic overflow.

A further proposal was to adopt a different scheme for handling arithmetic overflow; a simple alternative to the 2's complement overflow scheme is saturation arithmetic. In effect, in the model formulated above the function f is redefined so that $f(x) = x$ for $-1 < x < 1$ as before, but $f(x) = -1$ when $x \leq -1$ and $f(x) = 1$ for $x \geq 1$. With this system, Ebert *et al.* (1969) show that overflow oscillations cannot occur, and demonstrate that an initial nonzero state tends to zero at least as fast as for the equivalent linear system. Results of a similar kind for a generalised family of saturation arithmetic schemes appear in Ebert *et al.* (1969) and Willson (1972). However, saturation arithmetic does not provide the complete solution: Mitra (1977) demonstrates that overflow oscillations may nevertheless occur in any direct-form filter of third (or higher) order.

One consideration is that different realisations for a filter (i.e. arrangements of multipliers and addition units), all of which are equivalent under the linear model, can have different properties under the nonlinear regime. Another technique put forward for eliminating overflow oscillations is to employ a more sophisticated arrangement of components in order to achieve a realisation that in operation avoids the unwanted behaviour. Details of realisations known to be free from overflow oscillations are given in Barnes & Fam (1977); Mills *et al.* (1978) and the references therein. Further information on the different types of filter realisations and their relative merits may be found in Jackson *et al.* (1968), and in the

standard books on digital filtering, including Rabiner & Gold (1975) and Roberts & Mullis (1987).

The topic was revisited in 1988, by Chua and Lin, who approached it from a quite different point of view, regarding the nonlinear behaviour not as troublesome and something to be suppressed, but as interesting in its own right. They set $x_1(n) = y_{n-1}$, $x_2(n) = y_n$, so that the second order recurrence (2.31) describing the operation of the digital filter with a 2's complement overflow rule is rewritten as the coupled first order system

$$\begin{aligned}x_1(n) &= x_2(n-1) \\x_2(n) &= f(bx_1(n-1) + ax_2(n-1)),\end{aligned}\tag{2.33}$$

and the evolution of the output of the digital filter may be viewed as a sequence of iterates

$\mathbf{x}(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \end{pmatrix}$ under the nonlinear map $\mathbf{x} \mapsto \mathbf{x}'$ defined by

$$\begin{aligned}x'_1 &= x_2 \\x'_2 &= f(bx_1 + ax_2).\end{aligned}\tag{2.34}$$

(In the literature this map is frequently referred to as the state transition map, since it relates the current state of the filter, that is, the values stored in the two delay units, to the state at the previous time step.) Through this reformulation the focus is shifted from the practical design of digital filters to an investigation of the dynamics of a nonlinear system. It is thus no longer necessary for the multiplier values a and b to lie within the triangular region $b > -1$, $b < a+1$, $b < -a+1$ essential for stable operation in conventional digital filtering applications; instead a and b are regarded as parameters of the nonlinear map, and allowed to assume any real value. In their 1988 paper, Chua and Lin direct their attention to the choice of parameter values $b = -1$, $-2 < a < 2$ (the lower boundary of the aforementioned triangular region), for which the map (2.34) is precisely the nonlinear map \mathbf{F} defined in Section 2.1.1.

2.2.2 Literature pertaining to admissible periodic sequences

We summarise in this section the state of knowledge regarding admissible periodic sequences as it was when we began the work encompassed within this thesis. The presentation here is brief because most of the results will be mentioned again when we set them in the context of our investigations. We refer back to Section 2.1 for a discussion of the theory presented in Chua & Lin (1988), in which they link the dynamical behaviour of the nonlinear map to the notion of a symbolic sequence, and draw attention to the importance of admissibility for periodic sequences in describing the orbit structure of the state-space. Some further work, concerned with elucidating the nature of the orbits, is to be found in Chua & Lin (1988) and Davies & Sriranjana (1989), each includes computer plots of orbits for various initial points under the nonlinear map.

Several authors report the results of computer searches for short-period admissible sequences : Chua & Lin (1988, 1990b) and Wu & Chua (1993) both adopt a direct method of exhaustively testing strings via the criterion implicit in Proposition 2.1, and thereby obtain a list of the strings that generate admissible periodic sequences of lengths up to 25, but only when the parameter value is $a = 0.5$. Galias & Ogorzalek (1992) develop a different approach based on the exclusion of certain substrings, and are able to extend their search results up to strings of length 56 (again, only for $a = 0.5$).

Very few theoretical results concerning the admissibility of periodic sequences appear in the literature. Chua & Lin (1990b) employ their inequalities to show that certain families of sequences, generated by strings having a simple and regular pattern, do not generate admissible periodic sequences. Their results apply for a fixed parameter value $a = 2 \cos \theta$, and they show that the following strings of length N do not generate admissible periodic sequences :

- $+0 \dots 0, -0 \dots 0$: excluded for all even N and for sufficiently large odd N ,
- $++0 \dots 0, --0 \dots 0$: excluded for all odd N and for sufficiently large even N ,
- $+ - 0 \dots 0, - + 0 \dots 0$: excluded for sufficiently large N ,
- $+0 \dots 0 - 0 \dots 0$: excluded for sufficiently large N (equal runs of zeros).

Using an approach based on a geometrical construction of the orbits, Davies (1992) extended the knowledge about the strings $+0\dots 0$ and $-0\dots 0$, demonstrating that they do generate admissible periodic sequences when the parameter θ is below $\frac{\pi}{N}$. Davies gives an illustration that at a parameter value $\theta \approx 0.66$, sequences of this form account for several of the largest elliptical islands in state-space.

In addition to these results on the admissible periodic sequences, some work has been done to establish that particular finite length strings may not appear within any admissible sequence, at least for certain parameter values. Wu & Chua (1993) give conditions on the parameter so that the strings $++$, $--$, $+0+$, $-0-$, $+0-$ and $-0+$ are forbidden from appearing within any admissible sequence, and in a slightly more general result, Galias & Ogorzałek (1992) show that an admissible periodic sequence may not simultaneously include the strings $+0\dots 0+$ and $-0\dots 0+$ (of equal length).

2.3 A survey of related investigations

2.3.1 The nonlinear map F as part of a larger two-parameter family

It was shown in Section 2.2.1 that the nonlinear map F , the investigation of whose dynamics via the admissible periodic sequences forms the central theme of this thesis, originates within a larger family of maps modelling digital filter operation in the presence of arithmetic overflow. The maps of this larger family are

$$F_{a,b} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} x_2 \\ f(bx_1 + ax_2) \end{pmatrix} \quad \text{where } f(x) = ((x + 1) \bmod 2) - 1, \quad (2.35)$$

are parametrised by the real valued quantities a and b , and encompass our F as the subfamily $b = -1$, $-2 < a < 2$. In this section we survey the research that has been conducted to understand the behaviour of this larger family of maps. To do so, it is appropriate to divide up the (a, b) parameter space into six regions, within which qualitatively distinct types of behaviour are encountered. In fact, these six regions essentially relate to the different kinds

of behaviour found for the dynamical systems associated with the various corresponding linear maps $\mathbf{x} \mapsto \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix} \mathbf{x}$. Recall that the local behaviour of each nonlinear map $\mathbf{F}_{a,b}$ is described, at least away from the discontinuities, by its associated linear map. Because $\det \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix} = -b$, the map $\mathbf{F}_{a,b}$ is area preserving when $b = \pm 1$ (Regions 2 and 4), shrinks area when $|b| < 1$ (Regions 1 and 5) and expands area when $|b| > 1$ (Region 6).

For the six regions demarcated we show, in Figure 2.7, a computer plot of a single orbit, in each instance chosen to give some indication of the kind of dynamical behaviour encountered when the parameter values are drawn from that region. Associated with the plot of each orbit in state-space is a graphical depiction of the corresponding region from the parameter plane. For other computer simulations of orbits, we refer the reader to Kocarev (1995a,b), Kocarev *et al.* (1996) and Davies (1995).

In addition to the broad discrimination between various kinds of dynamical behaviour via correlation with these six regions of the parameter plane (each is precisely described below), there is a separate, and important, collection of special cases occurring when the parameters a and b are integer valued. For such parameter values, the map may be advantageously viewed as defined on a torus, by identifying the upper and lower, and the left and right, boundaries of I^2 . Through this the discontinuity is removed, and consequently the map, now a smooth map of the torus and with constant derivative, is very much simpler to study. Indeed the behaviour of these particular maps, and of more general families to which they belong, has been much investigated in the mainstream literature of dynamical systems: we mention some key features in the discussion on Region 4 below.

It is appropriate to remark here that relatively little is known about the behaviour of $\mathbf{F}_{a,b}$ throughout many of the parameter regions; certainly nothing approaching the completeness of the theory given in Section 2.1.

Region 1: ($b > -1$, $b < -a + 1$, $b < a + 1$; linear stability region for the digital filter)

When the parameter values lie inside the square $|a| + |b| < 1$, overflow cannot occur, so the behaviour of the nonlinear map is identical to that for the linear map $\mathbf{x} \mapsto \begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix} \mathbf{x}$. Iterates of any initial point asymptotically approach the origin, spiralling towards it if the matrix

$\begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$ has complex eigenvalues, or approaching it in the direction of an eigenvector if the eigenvalues are real.

For parameter values outside this central square, overflow does occur. Davies & Petkov (1995) present an empirical investigation, demonstrating a change in the nature of the orbits as the parameter values a, b are moved away from the lower boundary of the region; of course this boundary represents the case treated in Section 2.1. They show, by example, that the elliptical orbits occurring when the parameter values are on this boundary, give way to orbits that asymptotically spiral towards periodic points. A symbolic sequence of the digits $+$, 0 , $-$ may be associated to the orbit of a point in the same manner as in Section 2.1.1 (but now $s(\mathbf{x}) = -1$ if $bx_1 + ax_2 \geq 1$, etc.). It seems to be the case that the orbits of points which give rise to periodic sequences asymptotically approach the periodic points whose locations are calculated from the linear equations (2.3) with \mathbf{A} revised accordingly. Bose & Chen (1991) and Davies & Petkov (1995) take in turn a number of short sequences and for each use these linear equations to ascertain conditions on the parameters a and b for there to exist some periodic point giving rise to the sequence.

However, an analogue of the theory from Section 2.1, relating the dynamics of the map to the admissible periodic sequences, has not so far been developed for parameter values in Region 1, and it is perhaps very much more difficult to do so. One particular difference is that the set of points whose orbits asymptotically approach a particular periodic point, the analogue of one of our elliptical sets (Proposition 2.2), now has a complex structure, seemingly fractal in nature. Davies & Petkov (1995) include some computer-generated diagrams of these regions; for an indication of the variety and subtlety of the phenomena which may be anticipated here, see Mira *et al.* (1994), Mira (1996).

Region 2 : ($b = -1$, $-2 < a < 2$; lower boundary of the linear stability region)

This is the region where the maps $\mathbf{F}_{a,b}$ subsume our map \mathbf{F} ; results about its dynamical behaviour and their connection with admissible periodic sequences were described in Section 2.1, and results about the sequences themselves in Section 2.1.2. Here we outline work concerned with that part of the dynamics associated with points giving rise to aperiodic

sequences. Wu & Chua (1993) demonstrate the existence of points giving rise to aperiodic sequences (although no such point has been exhibited explicitly), and show that all the points giving rise to a particular aperiodic sequence must lie on a line segment in I^2 . Ashwin *et al.* (1997) and Kocarev *et al.* (1996) consider the set of points giving rise to all the aperiodic sequences. Ashwin does computer calculations to estimate the size of this set, and his results lead him to the conjecture that it has positive measure when the parameter θ is not a rational multiple of π . On the other hand, in the case of one of these excluded values ($\theta = \frac{\pi}{4}$, $a = \sqrt{2}$) he is able to calculate its Hausdorff dimension precisely, and thereby can show that the set has measure zero. Ashwin (1996) makes a start on the study of the dynamics of the restriction of \mathbf{F} to this set, and provides evidence that no orbit can be dense within it.

Region 3 : ($b = -a + 1$ or $b = a + 1$; reduction of dimension possible)

For parameter values on these two lines in the (a, b) plane, Kocarev *et al.* (1996) show that the orbit of any point lies on a one-dimensional subset of the state-space : this subset consists of a finite collection of parallel lines in I^2 . When the state-space is identified with the cylinder $I \times S^1$, in the case $b = -a + 1$ the parallel lines constitute a single helical arc, and in the case $b = a + 1$ two helical arcs with the iterates alternately visiting each. A reduction of dimension can thus be achieved by treating $\mathbf{F}_{a,b}$ as a family of one-dimensional maps, dependent on the parameter values and the initial point. In the special case $a = 2$, $b = -1$, the one-dimensional action of the map $\mathbf{F}_{a,b}$ is simply rotation on a circle, if the state-space is identified with a torus $S^1 \times S^1$; again, when $a = -2$, $b = -1$ the one-dimensional action of the maps $\mathbf{F}_{a,b}^2$ is rotation on a circle.

Region 4 : ($b = -1$, $|a| > 2$, or $b = 1$; map preserves area and has locally expanding and contracting directions)

We discuss first the special situation that arises for a parameter choice in this region but when a is an integer. Then if the state-space is identified with the torus $S^1 \times S^1$, the map $\mathbf{F}_{a,b}$ belongs to a well known class of maps, the hyperbolic toral automorphisms. A standard theory exists for these maps, using the construction of a Markov partition to establish a conjugacy between the nonlinear map and a shift map on a symbolic dynamical system,

a subshift of finite type. (This construction is entirely distinct from the association of a symbolic sequence to an orbit as introduced in Section 2.1, and used throughout the thesis.) Known properties of the hyperbolic toral automorphisms imply the following results for the map $F_{a,b}$:

- (a) The number of periodic points of period N is $\lambda^N + \lambda^{-N} - 2$, where λ is the larger of the two eigenvalues of the matrix $\begin{pmatrix} 0 & 1 \\ b & a \end{pmatrix}$. The periodic points are dense in I^2 .
- (b) $F_{a,b}$ is topologically mixing.
- (c) $F_{a,b}$ is ergodic with respect to Lebesgue measure; in particular the iterates of almost all initial points are uniformly distributed over I^2 .

A treatment of this material, and further information on hyperbolic toral automorphisms, is contained in Devaney (1989), and in Katok & Hasselblatt (1995). For the particular case when the parameter values are $b = -1$, $a = 3$ the nonlinear map $F_{a,b}$ identifies, under a linear change of coordinates, with the "cat" map of Arnol'd & Avez (1967).

Some of these results have been shown to transfer to the case when a is not an integer, i.e. to the situation when $F_{a,b}$ is not continuous (interpreted as a map on the torus). Kocarev & Chua (1993) cite work of Vienti (1992) which shows that the discontinuous sawtooth map, induced on the torus by the linear map of the plane with matrix $\begin{pmatrix} 1 & K \\ 1 & 1+K \end{pmatrix}$, is ergodic with respect to Lebesgue measure; in the case when $b = -1$, the map $F_{a,b}$ corresponds to this sawtooth map via $K = a - 2$ and the linear conjugation $\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & K \\ 1 & 1+K \end{pmatrix}$. Figure 2.7(d) accords with this, the iterates have the appearance of a uniform distribution throughout the state-space.

Region 5 : ($|b| < 1$, $b > -a + 1$, $b > a + 1$; map shrinks area and has local expanding and contracting directions)

Computational simulation of the orbits, undertaken by Kawamata *et al.* (1995), suggests that the map $F_{a,b}$ has an attractor for parameter values in this region; iterates of a randomly chosen initial point quickly approach some invariant set. On the evidence of these empirical investigations, the form of this attractor appears to be, locally, the product of a Cantor set

with a line segment, suggestive of the type of structure observed in attractors for other two-dimensional nonlinear maps (see for example Devaney (1989), Lozi (1978), Hénon (1976)). Two studies to calculate the dimension of the attracting set have been reported. Kawamata *et al.* (1995) use box counting techniques to compute an estimate for the dimension of the attracting set and thereby to investigate its dependence on the parameters a and b . Employing a different approach, Kocarev *et al.* (1996), apply work of Belykh (specific references are given in their paper) to obtain a theoretical upper bound for its dimension.

Region 6 : ($|b| > 1$; map expands area)

Very little research beyond the computer simulation of orbits has been conducted in this case. This region in the (a, b) plane can be further decomposed (see Kocarev *et al.* (1996)) into parts where the linear map associated with $F_{a,b}$ has either one expanding and one contracting direction, or two expanding directions. Kocarev *et al.* (1996) initiate the study, from a measure theoretic viewpoint, of some special instances of the map $F_{a,b}$ for parameters in this region, when at least one is an integer. Petkov & Davies (1997) extend their searches for periodic points, mentioned in the discussion of Region 1, and determine regions of the (a, b) plane, extending into Region 6, where the parameters allow for simple periodic orbits in state-space.

We end this survey by mentioning that there are results from dynamical systems theory, of a more abstract and general nature, that have application to the map $F_{a,b}$. The existence of invariant measures, absolutely continuous with respect to the Lebesgue measure, for a much more general classes of maps *with discontinuities* is established, in the one-dimensional case by Lasota & Yorke (1973), and in the higher dimensional case by Sataev (1992). For the existence of attractors, Kocarev *et al.* (1996) cite work of Pesin on attractors of hyperbolic maps with discontinuities. We remark though, that the map $F_{a,b}$ is simple in some respects : in particular, away from the discontinuities its derivative is constant. It therefore seems plausible that direct constructive proofs of these results could be found, as has been done for other simple maps; in this connection we mention Misiurewicz (1980) (a proof that the Lozi map has an attractor), Belykh (1995) (existence of attractors for a family of maps not unlike $F_{a,b}$), and Vaienti (1992) (ergodicity of the sawtooth map).

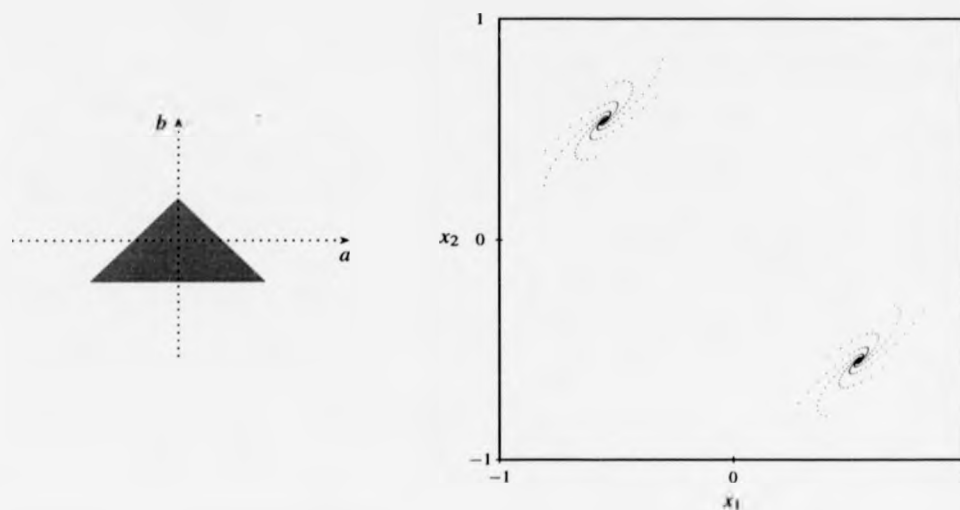
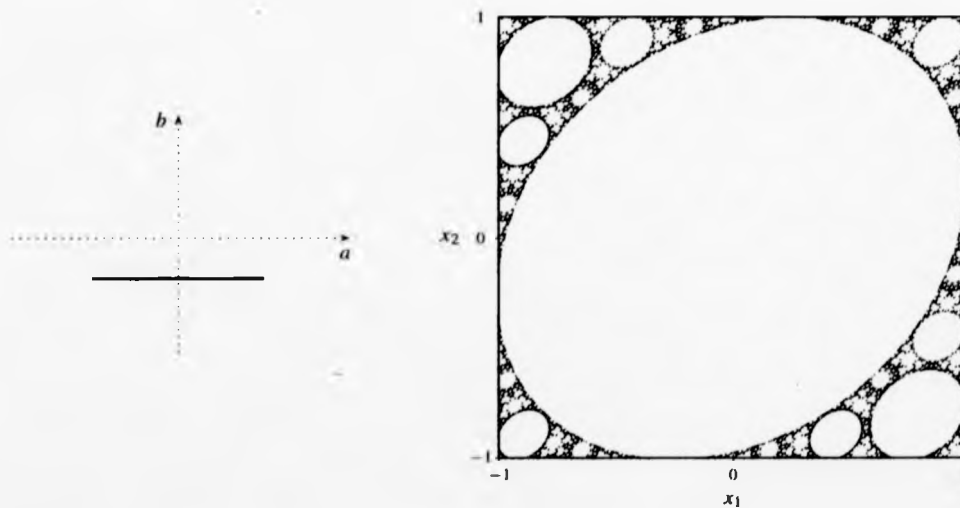
(a) Region 1 ($a = 1.7, b = -0.98$)(b) Region 2 ($a = 0.5, b = -1$)

Figure 2.7: The six parameter regions, and for each a typical orbit. The initial point is $(x_1, x_2) = (0.4, -0.8)$ always, and the choice of parameter values is indicated.

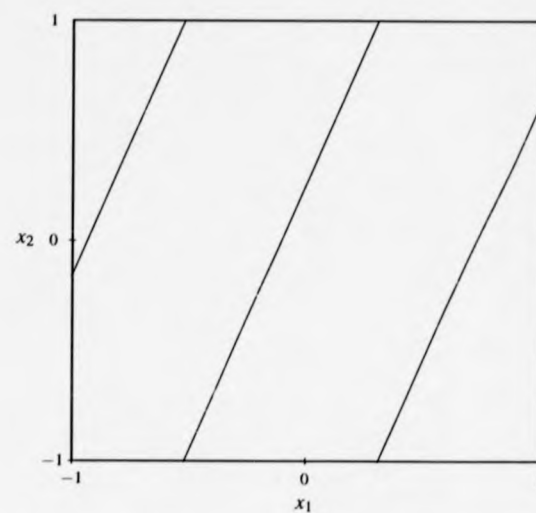
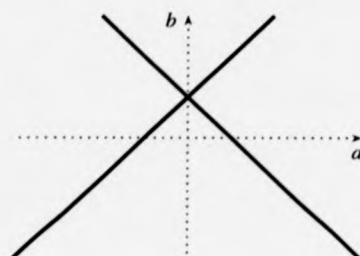
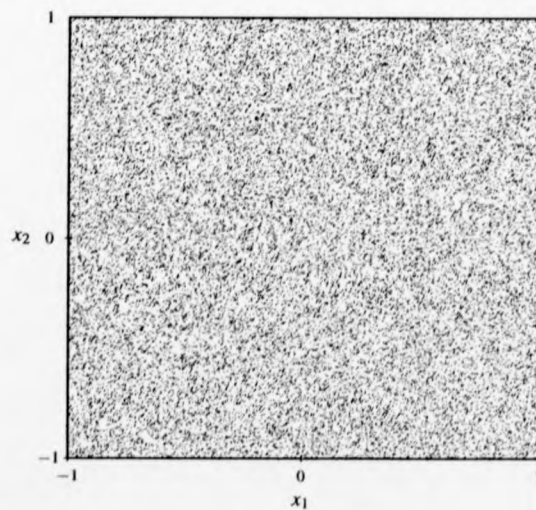
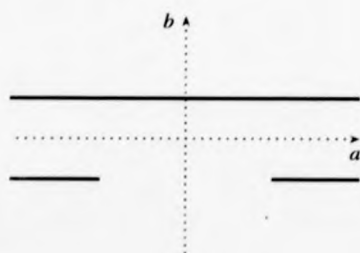
(c) Region 3 ($a = 3.4$, $b = -2.4$)(d) Region 4 ($a = 3.4$, $b = -1$)

Figure 2.7: continued.

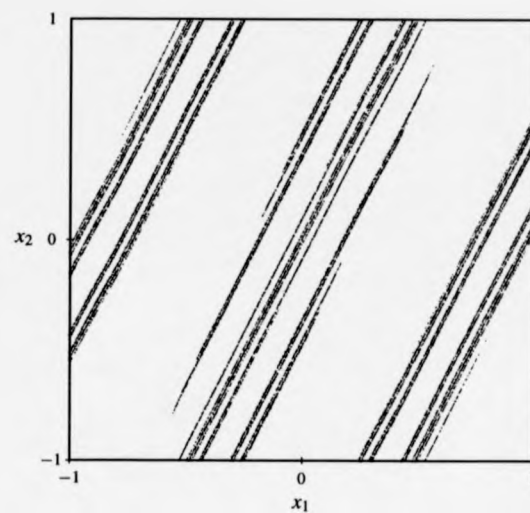
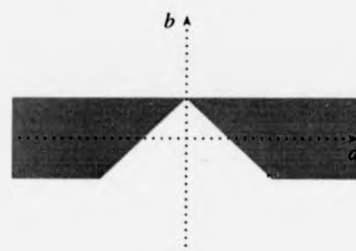
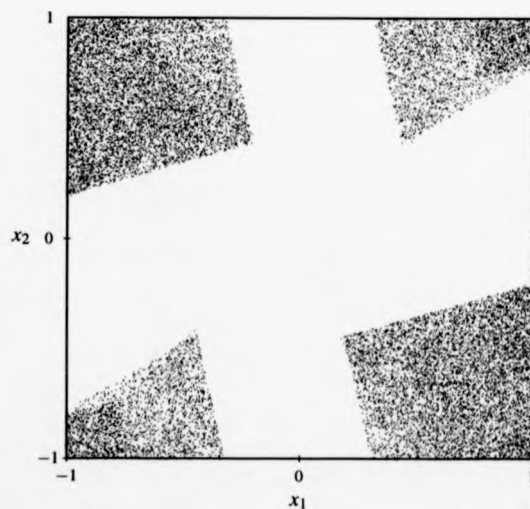
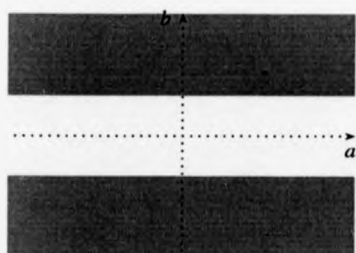
(e) Region 5 ($a = 2.3, b = -0.5$)(f) Region 6 ($a = 0.3, b = -1.15$)

Figure 2.7: continued.

2.3.2 Other topics

In this section we mention three further topics loosely connected with our area of study.

Higher Order Filters

Some investigations have been undertaken into the properties of nonlinear maps arising from other types of digital filter. Chua & Lin (1990a) initiate a study of a third order filter, with nonlinearity due to 2's complement arithmetic overflow. The three-dimensional map corresponding to this system is

$$\mathbf{F}_{a,b,c} : I^3 \rightarrow I^3, \quad \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} x_2 \\ x_3 \\ f(cx_1 + bx_2 + ax_3) \end{pmatrix},$$

now with three real-valued parameters a, b, c . Chua and Lin restrict attention to parameter values on one edge of the region for linear stability, where the behaviour of the map is similar to that found in Region 2 above. They show that the orbit of any point is confined to a number of parallel planar sheets, and thereby reduce the system to a two-dimensional one. In a manner similar to that for the map \mathbf{F} , they associate a symbolic sequence to the orbit of each point, but 7 symbols are needed now in place of the $+$, 0 , $-$ that served previously. They derive analogues of the results in Section 2.1, to describe the orbits associated with periodic sequences. One interesting difference is that the order in which the iterates visit these planar sheets is also determined by the symbolic sequence. Periodicity then imposes an additional requirement on the string $s_0 s_1 \dots s_{N-1}$ in order for it to generate an admissible periodic sequence, namely $s_0 + s_1 + \dots + s_{N-1} = 0$.

A second set of parameter values at which a reduction to two dimensions can be effected is explored in Mitrovski & Kocarev (1996), and the authors also consider the extension to higher order systems. Wu & Chua (1994) transfer some of their results for the second order filter map with parameters from Region 2 to systems of arbitrary order. Computer simulations of orbits of the third order filter map, for a variety of parameter values, including choices other than those for which the system is effectively two-dimensional, are given in Kelber (1995).

Maps deriving from alternative arithmetic systems

A number of studies have been reported to do with the behaviour of the nonlinear map resulting from a second order digital filter when overflow is handled by saturation arithmetic (cf. Section 2.2.1). Within the region of linear stability the behaviour is identical to the linear model. For a large part of the remainder of the (a, b) plane the behaviour is very simple, and iterates of any point eventually coincide with fixed or period-2 points located on the boundary of state-space. For parameter values elsewhere, iterates eventually enter and remain within a one-dimensional subset of state-space, namely the boundary of a convex polygon. Investigations concentrate on the behaviour of the associated one-dimensional map, whose domain is the boundary of this polygon, and also on the classification of the regions within the (a, b) plane where the map may have periodic points of various periods. Details of this work can be found in Ogorzałek (1992), Galias (1995), and Galias & Ogorzałek (1995).

We should perhaps anticipate fewer qualitatively different types of behaviour than for the 2's complement case because anything displaced by the linear map beyond the state-space is annihilated as opposed to being cycled back into the system.

Some investigations have also been carried out into the behaviour of the nonlinear maps for other arithmetic systems, see for example Rakhmanov *et al.* (1996), Rakhmanova *et al.* (1994).

Potential Applications

We conclude by mentioning briefly here some studies concerned with possible engineering applications arising out of various investigations on the behaviour of the nonlinear maps associated with digital filters. Interest concentrates on the use of the filters for signal generation, in view of the extremely wide range of signals that can be created from what in practice is a very simple system to implement. An indication of the variety of signals attainable may be derived from the plots of orbits in Figure 2.7. The map parameters a, b may be interpreted as control parameters for the selection of a desired type of signal (see Götz *et al.* (1995)). Investigations into the statistical properties of the signals that can be

achieved are found in Kutzer *et al.* (1994) and Kelber (1995).

That the filters might be employed for random number generation, because of the uniform distribution of iterates occurring when the parameter values are in Region 4, is discussed in Lin & Chua (1993). In that paper, and in Lin & Chua (1991), the authors take into consideration the effect of working to a finite arithmetic precision, constrained by word-length, instead of with real numbers.

Götz *et al.* (1996) and Petkov & Davies (1996) consider the possibility of using the signals from digital filters as the basis for secure communication systems, and this topic is one of considerable current interest. Of course, with this kind of application in mind, a sound theoretical understanding of the maps is essential to avoid the possibility that the output signals are simpler than communication designers believe them to be, a situation exposing a supposedly secure system to attack.

2.4 Elementary inferences from the Chua-Lin inequalities

As a final preparation, before embarking upon our major contributions in the following chapters, we collect together here, in Proposition 2.4, five basic permanence properties about strings that satisfy the Chua-Lin inequalities. Although most of what we include has been observed previously, at least in some equivalent form, the various results appear widely scattered in the literature, and it is convenient to collect and systematise them, primarily by way of reference for their use later in the thesis. Sources are Wu & Chua (1993, Remark 1), Kocarev *et al.* (1996, Theorems 4 and 8), Ebert *et al.* (1969).

Proposition 2.4

1. (Negation of every digit)

If the string $s_0s_1 \dots s_{N-1}$ satisfies the Chua-Lin inequalities, then so does the string $\bar{s}_0\bar{s}_1 \dots \bar{s}_{N-1}$.

2. (Cycling)

If the string $s_0s_1 \dots s_{N-1}$ satisfies the Chua-Lin inequalities, then so does the string $s_1s_2 \dots s_{N-1}s_0$.

3. (String reversal)

If the string $s_0 s_1 \dots s_{N-1}$ satisfies the Chua-Lin inequalities, then so does the string $s_{N-1} s_{N-2} \dots s_1 s_0$.

4. (Multiple copies)

A string $s_0 s_1 \dots s_{N-1}$, consisting of h concatenated copies of $s_0 s_1 \dots s_{m-1}$, satisfies the Chua-Lin inequalities precisely when $s_0 s_1 \dots s_{m-1}$ does.

5. (Negation of alternate digits)

If the string $s_0 s_1 \dots s_{N-1}$, where N is even, satisfies the Chua-Lin inequalities at the parameter value θ then so does the string $s_0 \bar{s}_1 s_2 \bar{s}_3 \dots s_{N-2} \bar{s}_{N-1}$ at the parameter value $\pi - \theta$.

(In 1, 2, 3 and 4 it is to be understood that the string and its transformed version are considered at the same parameter value. Also if the digit s_r is $+$, 0 or $-$ then the digit \bar{s}_r represents $-$, 0 or $+$ respectively.)

Proof

1 and 2 follow immediately from the Chua-Lin inequalities.

To keep the presentation simple the reasoning presented below for 3, 4 and 5 is set out only for f_0 , the function appearing in the first Chua-Lin inequality. In each case the generalisation to the other functions f_r follows simply by cycling.

3. For the string $s_0 s_1 \dots s_{N-1}$, the expression for the function f_0 is

$$\begin{aligned} f_0 &= s_0 \cos \frac{N}{2} \theta + \sum_{i=1}^{N-1} s_i \cos \frac{N-2i}{2} \theta \\ &= s_0 \cos \frac{N}{2} \theta + \sum_{i=1}^{N-1} s_{N-i} \cos \frac{N-2i}{2} \theta, \end{aligned} \quad (2.36)$$

because $\cos \frac{N-2(N-i)}{2} \theta = \cos \frac{N-2i}{2} \theta$. Hence f_0 for the string $s_0 s_1 \dots s_{N-1}$ is identical to the function f_{N-1} when formulated for the reversed string $s_{N-1} \dots s_1 s_0$.

4. Consider the expression for f_0 for the longer string $s_0 s_1 \dots s_{N-1}$, consisting of h copies of $s_0 s_1 \dots s_{m-1}$,

$$\sum_{r=0}^{N-1} s_i \cos \frac{N-2i}{2} \theta = \sum_{i=0}^{m-1} s_i \left(\sum_{j=0}^{h-1} \cos \frac{N-2(jm+i)}{2} \theta \right), \quad (2.37)$$

and using the expression for the sum of the cosine series,

$$\begin{aligned} \cos(pa+b) + \cos((p+1)a+b) + \dots + \cos(qa+b) = \\ \frac{\sin \frac{1}{2}(q-p+1)a}{\sin \frac{1}{2}a} \cos \left(\frac{1}{2}(p+q)a+b \right) \end{aligned} \quad (2.38)$$

we have

$$\begin{aligned} \sum_{j=0}^{h-1} \cos \frac{N-2(jm+i)}{2} \theta &= \sum_{j=0}^{h-1} \cos(jm\theta + (i - \frac{N}{2})\theta) = \frac{\sin \frac{1}{2}hm\theta}{\sin \frac{1}{2}m\theta} \cos \left(\frac{1}{2}(h-1)m\theta + (i - \frac{N}{2})\theta \right) \\ &= \frac{\sin \frac{1}{2}N\theta}{\sin \frac{1}{2}m\theta} \cos \left(-\frac{1}{2}m\theta + i\theta \right) = \frac{\sin \frac{1}{2}N\theta}{\sin \frac{1}{2}m\theta} \cos \frac{m-2i}{2} \theta. \end{aligned} \quad (2.39)$$

This means that

$$\frac{\sum_{i=0}^{N-1} s_i \cos \frac{N-2i}{2} \theta}{\sin \frac{N}{2} \theta \sin \theta} = \frac{\sum_{i=0}^{m-1} s_i \cos \frac{m-2i}{2} \theta}{\sin \frac{m}{2} \theta \sin \theta}, \quad (2.40)$$

so the longer string satisfies the first inequality if and only if the shorter one does.

(Of course, from the point of view that some point in state-space gives rise to an admissible periodic sequence, the issue here is quite transparent, but the verification via the trigonometric formulae establishes consistency with the work that we subsequently engage in.)

5. The function f_0 formulated for the string $s_0 \bar{s}_1 s_2 \bar{s}_3 \dots s_{N-2} \bar{s}_{N-1}$ and evaluated at $\pi - \theta$ is

$$\begin{aligned} \sum_{i=0}^{N-1} (-1)^i s_i \cos \frac{N-2i}{2} (\pi - \theta) &= \sum_{i=0}^{N-1} (-1)^i s_i \cos \left(\frac{N}{2} - i \right) \pi \cos \frac{N-2i}{2} \theta \\ &= (-1)^{\frac{N}{2}} \sum_{i=0}^{N-1} s_i \cos \frac{N-2i}{2} \theta = (-1)^{\frac{N}{2}} f_0(\theta). \end{aligned} \quad (2.41)$$

□

Because of the connection between the admissible periodic sequences and the dynamics of the map F , each of the items of Propositions 2.4 has a counterpart in the geometry of the orbits in state-space. If the sequence generated by $s_0 s_1 \dots s_{N-1}$ is admissible, so that the iterates of any point giving rise to it visit in turn the N ellipses centred at the points $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_{N-1}$, then

- 1'. (**Negation**) The iterates of a point giving rise to $\bar{s}_0 \bar{s}_1 \dots \bar{s}_{N-1}$ visit in turn N ellipses centred on $-\mathbf{x}_0, -\mathbf{x}_1, \dots, -\mathbf{x}_{N-1}$; any point \mathbf{x} giving rise to $s_0 s_1 \dots s_{N-1}$ corresponds to the point $-\mathbf{x}$ giving rise to $\bar{s}_0 \bar{s}_1 \dots \bar{s}_{N-1}$.
- 2'. (**Cycling**) The iterates of a point giving rise to $s_1 s_2 \dots s_{N-1} s_0$ visit in turn the N ellipses centred on $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_{N-1}, \mathbf{x}_0$; any point \mathbf{x} giving rise to $s_0 s_1 \dots s_{N-1}$ corresponds to a point $F(\mathbf{x})$ giving rise to $s_1 s_2 \dots s_{N-1} s_0$.
- 3'. (**Reversal**) The iterates of a point giving rise to $s_{N-1} \dots s_1 s_0$ visit in turn the N ellipses centred on $\mathbf{M}\mathbf{x}_0, \mathbf{M}\mathbf{x}_{N-1}, \dots, \mathbf{M}\mathbf{x}_1$, where $\mathbf{M} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ swaps the coordinates of a point \mathbf{x} ; any point \mathbf{x} giving rise to $s_0 s_1 \dots s_{N-1}$ corresponds to a point $\mathbf{M}\mathbf{x}$ giving rise to $s_{N-1} \dots s_1 s_0$.
- 4'. (**Multiple copies**) As noted above, the orbits for the strings of length m and $N = hm$ are identical.

The justification of 1', 2' and 4' is straightforward. For 3', note first that it follows from

$$\mathbf{x}_0 = (\mathbf{I} - \mathbf{A}^N)^{-1} (s_0 \mathbf{A}^{N-1} + s_1 \mathbf{A}^{N-2} + \dots + s_{N-1} \mathbf{I}) \mathbf{b}$$

that

$$\mathbf{M}\mathbf{x}_0 = (\mathbf{I} - \mathbf{A}^N)^{-1} (s_{N-1} \mathbf{A}^{N-1} + s_{N-2} \mathbf{A}^{N-2} + \dots + s_1 \mathbf{A} + s_0 \mathbf{I}) \mathbf{b},$$

because $\mathbf{M}\mathbf{A} = \mathbf{A}^{-1}\mathbf{M}$, $(\mathbf{I} - \mathbf{A}^{-N})^{-1} = -(\mathbf{I} - \mathbf{A}^N)^{-1} \mathbf{A}^N$ and $\mathbf{A}\mathbf{M}\mathbf{b} = -\mathbf{b}$. Treating $\mathbf{M}\mathbf{x}_1, \dots, \mathbf{M}\mathbf{x}_{N-1}$ similarly, we find that the iterates of a point giving rise to $s_{N-1} \dots s_1 s_0$ visit in turn the N ellipses centred on $\mathbf{M}\mathbf{x}_0, \mathbf{M}\mathbf{x}_{N-1}, \dots, \mathbf{M}\mathbf{x}_1$. The symmetry properties inherent in 1' and 3', of the points giving rise to periodic sequences, account for the reflection

symmetries in the diagonals $x_1 = x_2$ and $x_1 = -x_2$ that is observed for the elliptical islands in Figure 2.1.

Throughout the thesis we will frequently wish to report lists of strings that generate admissible periodic sequences, and we use two of the permanence properties described in Proposition 2.4 to effect a sensible economy in the way these lists are presented. Firstly, there is no need to report any string that is *reducible*, in the sense that it is a multiple copy of a substring, since the sequence it generates can be accounted for via some shorter string embedded within it. Indeed we need only list each *irreducible* string, that is a string not a multiple copy of one of its substrings. Secondly we choose to display just one of the N cycled versions of any string that generates a periodic sequence of least period N because, although each of the N cycled versions do, in a strict sense, generate distinct periodic sequences, there is a trivial relationship between these sequences; given any two such, either may be shifted so that they match where they overlap. At times where it is appropriate we will consider periodic sequences related in this way to be equivalent, and to define essentially a single sequence.

Having decided to identify by a single irreducible representative all the strings that are related by cycling or concatenation of multiple copies, it is significant to ask when each of the other transformations of Proposition 2.4 produces a string whose periodic sequence is distinct from that of the original, untransformed, string. In the case of the first two transformations, i.e. negation of every digit and string reversal, we state the general answer, and provide examples of strings that do and do not produce a new sequence. The justification in these two cases follows simply from the requirement that some cycled version $s_r s_{r+1} \dots s_{r-1}$ matches the original string.

- (a) **Negation of every digit :** The string got from $s_0 s_1 \dots s_{N-1}$ by negating each digit produces a new periodic sequence except when $s_0 s_1 \dots s_{N-1}$ is of even length and has the form $s_0 s_1 \dots s_{\frac{N}{2}-1} \bar{s}_0 \bar{s}_1 \dots \bar{s}_{\frac{N}{2}-1}$.

Examples :

1. **+00-00** does not produce a new sequence under negation.

2. No cycled version of the negation of $+00+$ matches the original string.

(b) **String reversal** : The string produced by reversing $s_0s_1 \dots s_{N-1}$ generates a new periodic sequence unless it can be split into two component substrings, $s_0s_1 \dots s_{r-1}$ and $s_rs_{r+1} \dots s_{N-1}$, both of which are palindromic. (We remark that decomposability of a string in this way is inherited by each cycled version of the string.)

Examples :

1. $0++0-0-$, with palindromic components $0++0$ and $-0-$, does not produce a new sequence under reversal.
2. No cycled version of the reversal of $++00+0$ matches the original string.

The final transformation of Proposition 2.4, namely the negation of alternate digits of a string of even length, is rather different because here we relate two strings which generate periodic sequences admissible at *different values of θ* . Thus we obtain new information even when the strings are equivalent in the sense described above. One additional point to be aware of for this transformation is that two strings related through the negation of alternate digits need not generate periodic sequences of the same (least) period (i.e. irreducibility may not be preserved). Consideration of the strings $++0000$ and $+00-00$ illustrates the two situations that can arise :

$++0000 \xleftrightarrow{\text{alternate negation}} +-0000$: both strings generate a period-6 sequence

$+00-00 \xleftrightarrow{\text{alternate negation}} +00+00$: $\begin{cases} +00-00 \text{ generates a period-6 sequence} \\ +00+00 \text{ generates a period-3 sequence} \end{cases}$

The general result here is that the string produced by negating alternate digits of the irreducible string $s_0s_1 \dots s_{N-1}$, with N even, generates a sequence of least period N , except in the case when $\frac{N}{2}$ is odd and the original string has the form $s_0s_1 \dots s_{\frac{N}{2}-1}s_0s_1 \dots s_{\frac{N}{2}-1}$, when it generates a sequence of least period $\frac{N}{2}$. This is established by the following three steps :

- (i) If $s_0s_1 \dots s_{N-1}$ transformed consists of h copies of an irreducible string of length m ,

then m must be odd, for otherwise

$$s_i = s_{m+i} = s_{2m+i} = \cdots = s_{(h-1)m+i}$$

for $0 \leq i \leq m-1$, and the original string would be reducible.

(ii) Because of (i), h is even; h cannot exceed 2, for otherwise

$$s_i = s_{2m+i} = s_{4m+i} = \cdots = s_{(h-2)m+i}.$$

(iii) The transformed string consists of two repeated halves when $(-1)^{\frac{N}{2}+i} s_{\frac{N}{2}+i} = (-1)^i s_i$,
i.e. precisely when $s_{\frac{N}{2}+i} = -s_i$.

Chapter Summary

- The connection between the dynamics of the nonlinear filter map and admissible periodic sequences is described; attention is drawn to the prominent role played by these sequences in elucidating the behaviour of the map.
- Some fundamental results presented are the criterion of Chua and Lin for the admissibility of a periodic sequence, a description of the orbits of points that give rise to periodic sequences, and the partitioning of state-space into subsets of points associated with the various admissible periodic sequences. (The discussion is more general than that in the published articles of Chua and Lin, here rational as well as irrational values of the map parameter are treated.)
- The criterion for a string to generate an admissible periodic sequence is recast as a system of trigonometric inequalities, following Chua and Lin (1990b). A version involving strict inequalities is fixed on as an appropriate starting point for the ensuing theoretical investigations of admissible periodic sequences.
- All previously published results concerning admissible periodic sequences are reported and reviewed. To supply a wider perspective on the issues pursued, there is also included a brief survey of recent research on the dynamics of a larger, two-parameter, family of maps arising in digital filters that includes the map F that is the subject of our studies.
- Some elementary, but useful, permanence properties of admissible periodic sequences are collected together; each is verified via the Chua-Lin inequalities. Considered are the operations on the generating string of digit negation, cycling of digits, reversal, concatenation of multiple copies, and negation of alternate digits.

Chapter 3

Strings which Appear Within Admissible Sequences

The string $+0+$ never appears within any admissible sequence regardless of the parameter value. In the context of how little is known about the nature of the admissible sequences, and of their evident complexity, this is a surprisingly sharp result. Motivated by it, we conduct a full study into when strings of the form $+0\dots 0+$ may appear within admissible sequences; our answer is comprehensive, and encompasses all earlier partial results.

Ultimately we wish to understand the admissible periodic sequences, and the conclusions we obtain here will later contribute towards that objective. The point is that we have the means to exclude from consideration those periodic sequences containing forbidden substrings. The idea can be used to good effect both in the theoretical elucidation of admissible periodic sequences and also as the basis of some of our computer algorithms employed in the computational search for these sequences.

3.1 The strings $+0\dots 0+$ as substrings of admissible sequences : dependence on the parameter θ

The main goal of this chapter is to answer the specific question "For which values of the parameter θ does a string $+0\dots 0+$ appear as a substring of some admissible sequence?"

We are able to answer this question completely, by way of Proposition 3.1 below. In general terms our result states that the values of θ , where a string $+0 \dots 0+$ of length N appears in some admissible sequence, belong to a finite union of between zero and $\frac{1}{2}N$ open intervals, and we give a precise description of these intervals.

Proposition 3.1

The string $+0 \dots 0+$ is a substring of some admissible sequence if and only if θ belongs to one of the following intervals : for each k such that $1 \leq k \leq \lfloor \frac{N}{2} \rfloor$ determine, where possible, the least positive *even* integers s_1, s_2 satisfying

$$ks_1 \equiv 1 \pmod{N+1} \quad \text{and} \quad ks_2 \equiv -1 \pmod{N+1}, \quad (3.1)$$

and an interval is

$$\left(\frac{2\pi}{N+1} \frac{ks_1-1}{s_1}, \frac{2\pi}{N+1} \frac{ks_2+1}{s_2} \right). \quad (3.2)$$

Remark : Implicit in the statement of this result is that at values of θ not belonging to any one of the intervals (3.2), the string *cannot* appear as a substring of *any* admissible sequence. In particular when N is odd the congruences (3.1) have no solution in even integers so the string cannot appear in any admissible sequence regardless of the value of θ .

The proof of Proposition 3.1 is not straightforward, and involves a long geometric argument which we present in Sections 3.2 to 3.8 of this chapter. Before embarking on that we present some examples which show how the intervals are calculated from the formulae (3.1) and (3.2).

Example 3.1 : Calculation of the intervals for the string $+000000+$

Here the string is of length $N = 8$, so k may take the values 1, 2, 3 and 4. To illustrate the calculations we derive the interval for $k = 2$, where the congruences (3.1) become

$$2s_1 \equiv 1 \pmod{9} \quad \text{and} \quad 2s_2 \equiv -1 \pmod{9}.$$

In such a simple example as this, where the congruence modulus is small, the easiest way

to find s_1 and s_2 is by repeated trial of the even integers 2, 4, 6, ..., until a solution is found. We obtain $s_1 = 14$ and $s_2 = 4$, and the corresponding interval is $(\frac{2\pi}{9} \cdot \frac{27}{14}, \frac{2\pi}{9} \cdot \frac{9}{4}) = (\frac{3\pi}{7}, \frac{\pi}{2})$.

The intervals for $k = 1$ and $k = 4$ may be determined in like manner, and are $(\frac{\pi}{3}, \frac{\pi}{4})$ when $k = 1$, and $(\frac{7\pi}{8}, \pi)$ when $k = 4$. The remaining value, $k = 3$, does not correspond to an interval, since in this case k and $(N + 1)$ share the common factor 3 and the congruences (3.1) have no solutions. Thus we conclude that the string **+00000+** appears as a substring of some admissible sequence when θ belongs to one of the intervals $(\frac{\pi}{3}, \frac{\pi}{4})$, $(\frac{3\pi}{7}, \frac{\pi}{2})$ or $(\frac{7\pi}{8}, \pi)$; for all other values of θ between 0 and π , the string **+00000+** does not appear in any admissible sequence.

Example 3.2 : Tabulating the intervals

For any particular string length N , the intervals (3.2) can be efficiently determined by computer. When k and $(N + 1)$ are coprime, Euclid's algorithm will produce integers m and n such that $km + (N + 1)n = 1$. Then m and $-m$ respectively satisfy the two congruences (3.1), and the least positive even solutions s_1 and s_2 are obtained by adding or subtracting a multiple of $(N + 1)$. This method was used to create Table 3.1, which lists, for string lengths $N \leq 30$, all the intervals for the parameter θ where a string **+0...0+** appears as a substring within some admissible sequence.

It is appropriate at this stage to mention earlier results specifying the parameter values for which certain types of string may appear within admissible sequences. Little has been established with any generality : the work in this area which has been previously published is summarised in Results 3.1 and 3.2 below.

Result 3.1 (Wu & Chua 1993)

The following strings cannot appear within admissible sequences for the parameter ranges indicated :

- (i) **++** and **--** over $0 < \theta < \frac{\pi}{2}$,
- (ii) **+0+** and **-0-** over $0 < \theta < \frac{3\pi}{4}$,
- (iii) **-0+** and **+0-** over $\frac{\pi}{3} < \theta < \frac{\pi}{2}$.

| String Length (N) | Intervals for θ |
|--------------------------|---|
| 2 | $(\frac{\pi}{2}, \pi)$ |
| 4 | $(\frac{\pi}{3}, \frac{\pi}{2}), (\frac{3\pi}{4}, \pi)$ |
| 6 | $(\frac{\pi}{4}, \frac{\pi}{3}), (\frac{\pi}{2}, \frac{3\pi}{5}), (\frac{5\pi}{6}, \pi)$ |
| 8 | $(\frac{\pi}{5}, \frac{\pi}{4}), (\frac{3\pi}{7}, \frac{\pi}{2}), (\frac{7\pi}{8}, \pi)$ |
| 10 | $(\frac{\pi}{6}, \frac{\pi}{5}), (\frac{\pi}{3}, \frac{3\pi}{8}), (\frac{\pi}{2}, \frac{5\pi}{9}), (\frac{5\pi}{7}, \frac{3\pi}{4}), (\frac{9\pi}{10}, \pi)$ |
| 12 | $(\frac{\pi}{7}, \frac{\pi}{6}), (\frac{3\pi}{10}, \frac{\pi}{5}), (\frac{5\pi}{11}, \frac{\pi}{2}), (\frac{3\pi}{5}, \frac{5\pi}{8}), (\frac{3\pi}{4}, \frac{7\pi}{9}), (\frac{11\pi}{12}, \pi)$ |
| 14 | $(\frac{\pi}{8}, \frac{\pi}{7}), (\frac{\pi}{4}, \frac{3\pi}{11}), (\frac{\pi}{2}, \frac{7\pi}{13}), (\frac{13\pi}{14}, \pi)$ |
| 16 | $(\frac{\pi}{9}, \frac{\pi}{8}), (\frac{3\pi}{13}, \frac{\pi}{4}), (\frac{\pi}{3}, \frac{5\pi}{14}), (\frac{7\pi}{15}, \frac{\pi}{2}), (\frac{7\pi}{12}, \frac{3\pi}{5}), (\frac{9\pi}{11}, \frac{5\pi}{6}), (\frac{15\pi}{16}, \pi)$ |
| 18 | $(\frac{\pi}{10}, \frac{\pi}{9}), (\frac{\pi}{5}, \frac{3\pi}{14}), (\frac{5\pi}{16}, \frac{\pi}{3}), (\frac{5\pi}{12}, \frac{3\pi}{7}), (\frac{\pi}{2}, \frac{9\pi}{17}), (\frac{5\pi}{8}, \frac{7\pi}{11}), (\frac{11\pi}{15}, \frac{3\pi}{4}), (\frac{5\pi}{6}, \frac{11\pi}{13}), (\frac{17\pi}{18}, \pi)$ |
| 20 | $(\frac{\pi}{11}, \frac{\pi}{10}), (\frac{3\pi}{16}, \frac{\pi}{5}), (\frac{3\pi}{8}, \frac{5\pi}{13}), (\frac{9\pi}{19}, \frac{\pi}{2}), (\frac{3\pi}{4}, \frac{13\pi}{17}), (\frac{19\pi}{20}, \pi)$ |
| 22 | $(\frac{\pi}{12}, \frac{\pi}{11}), (\frac{\pi}{6}, \frac{3\pi}{17}), (\frac{\pi}{4}, \frac{5\pi}{19}), (\frac{\pi}{3}, \frac{7\pi}{20}), (\frac{3\pi}{5}, \frac{7\pi}{16}), (\frac{7\pi}{15}, \frac{11\pi}{18}), (\frac{9\pi}{13}, \frac{7\pi}{10}), (\frac{7\pi}{9}, \frac{11\pi}{14}), (\frac{13\pi}{15}, \frac{7\pi}{8}), (\frac{21\pi}{22}, \pi)$ |
| 24 | $(\frac{\pi}{13}, \frac{\pi}{12}), (\frac{3\pi}{19}, \frac{\pi}{6}), (\frac{5\pi}{21}, \frac{\pi}{4}), (\frac{7\pi}{22}, \frac{\pi}{3}), (\frac{11\pi}{23}, \frac{\pi}{2}), (\frac{5\pi}{9}, \frac{9\pi}{16}), (\frac{7\pi}{11}, \frac{13\pi}{14}), (\frac{5\pi}{8}, \frac{15\pi}{18}), (\frac{7\pi}{7}, \frac{15\pi}{17}), (\frac{23\pi}{24}, \pi)$ |
| 26 | $(\frac{\pi}{14}, \frac{\pi}{13}), (\frac{\pi}{7}, \frac{3\pi}{20}), (\frac{5\pi}{17}, \frac{3\pi}{10}), (\frac{7\pi}{19}, \frac{3\pi}{8}), (\frac{\pi}{2}, \frac{13\pi}{25}), (\frac{13\pi}{22}, \frac{3\pi}{5}), (\frac{17\pi}{23}, \frac{3\pi}{4}), (\frac{13\pi}{16}, \frac{9\pi}{11}), (\frac{25\pi}{26}, \pi)$ |
| 28 | $(\frac{\pi}{15}, \frac{\pi}{14}), (\frac{3\pi}{22}, \frac{\pi}{7}), (\frac{5\pi}{24}, \frac{\pi}{5}), (\frac{3\pi}{11}, \frac{5\pi}{18}), (\frac{7\pi}{17}, \frac{5\pi}{12}), (\frac{13\pi}{27}, \frac{\pi}{2}), (\frac{13\pi}{20}, \frac{5\pi}{9}), (\frac{13\pi}{21}, \frac{5\pi}{8}), (\frac{11\pi}{16}, \frac{9\pi}{13}), (\frac{3\pi}{4}, \frac{19\pi}{25}), (\frac{19\pi}{23}, \frac{5\pi}{6}), (\frac{17\pi}{19}, \frac{9\pi}{10}), (\frac{27\pi}{28}, \pi)$ |
| 30 | $(\frac{\pi}{16}, \frac{\pi}{15}), (\frac{\pi}{8}, \frac{3\pi}{23}), (\frac{5\pi}{26}, \frac{\pi}{3}), (\frac{\pi}{4}, \frac{7\pi}{27}), (\frac{9\pi}{28}, \frac{\pi}{2}), (\frac{5\pi}{13}, \frac{7\pi}{18}), (\frac{9\pi}{20}, \frac{5\pi}{11}), (\frac{11\pi}{19}, \frac{7\pi}{12}), (\frac{11\pi}{14}, \frac{11\pi}{17}), (\frac{17\pi}{24}, \frac{5\pi}{7}), (\frac{17\pi}{22}, \frac{7\pi}{9}), (\frac{5\pi}{6}, \frac{21\pi}{25}), (\frac{9\pi}{10}, \frac{19\pi}{21}), (\frac{29\pi}{30}, \pi)$ |

Table 3.1: The intervals of θ values where strings $+0 \dots 0+$ appear within some admissible sequence, tabulated for strings of length $N \leq 30$. (When N is odd the string is forbidden for all $0 < \theta < \pi$.)

Result 3.2 (Galias & Ogorzałek 1992)

For a given parameter value θ the string $s_0 0 \dots 0 +$ of length N , where the digit s_0 is given by the formula

$$s_0 = \begin{cases} + & \text{if } \sin N\theta > 0 \\ - & \text{if } \sin N\theta < 0, \end{cases}$$

cannot appear within any admissible sequence.

With regard to the family of strings $+0 \dots 0+$, both of these are partial results in the sense that they specify certain combinations of the parameter θ and the string length N at which the string $+0 \dots 0+$ is ruled out (i.e. cannot appear in any admissible sequence), and both come under the scope of our general description given by Proposition 3.1. We will discuss (in Section 3.9) the other families of strings mentioned in Results 3.1 and 3.2. There we indicate how our methods can be extended to provide the corresponding information for all strings of the form $-0 \dots 0-$, $+0 \dots 0-$, and $-0 \dots 0+$, in addition to $+0 \dots 0+$.

3.2 A criterion to identify strings which can appear within admissible sequences

Our starting point for the determination of the range of θ value for which the string $+0 \dots 0+$ appears in an admissible sequence is to develop an idea introduced by Galias and Ogorzałek in their 1992 paper. They present a condition for deciding when a string $s_0 s_1 \dots s_{N-1}$ cannot appear as part of any admissible sequence (Theorem 1 of the paper), and afterwards use their result as the basis of a computational algorithm whose purpose is to identify those strings which, for a given fixed parameter value θ , cannot appear in any admissible sequence. We show here that this approach may be taken further, and in particular that the converse of their result also holds. So that in fact the condition of Galias and Ogorzałek specifies *precisely* whether or not a string $s_0 s_1 \dots s_{N-1}$ can appear as a substring within an admissible sequence.

Throughout this section we will work with the original state-space coordinates for the filter map, so as to keep our presentation in agreement with what appears in Galias & Ogorzałek (1992). In fact, the choice of coordinate system is not important for the proof of Proposition 3.2 below. (Subsequently, in the remainder of the present chapter and in Chapter 4, we will adopt whichever coordinate system is the most convenient; it is the case that for the investigation of the strings $+0 \dots 0+$ there is an advantage in making the coordinate change described in Section 2.1.2.) Recall that, in state-space coordinates, the filter map \mathbf{F} is studied in the domain $I^2 = [-1, 1) \times [-1, 1)$, and may be written as

$$\mathbf{F}(\mathbf{x}) = \mathbf{A}\mathbf{x} + s\mathbf{b}, \quad (3.3)$$

where

$$\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -1 & a \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}, \quad \text{and } s = \begin{cases} -1 & \text{if } -x_1 + ax_2 \geq 1 \\ 1 & \text{if } -x_1 + ax_2 < -1 \\ 0 & \text{otherwise,} \end{cases} \quad (3.4)$$

with $a = 2\cos\theta$ as the parameter.

Theorem 1 in Galias & Ogorzałek (1992) states that if the set

$$S = \mathbf{w}_0(I^2) \cap \mathbf{w}_1(I^2) \cap \dots \cap \mathbf{w}_N(I^2), \quad (3.5)$$

is empty then the string $s_0s_1 \dots s_{N-1}$ cannot appear in any admissible sequence. The functions \mathbf{w}_r , that are used here, are defined in terms of the string $s_0s_1 \dots s_{N-1}$ by

$$\begin{aligned} \mathbf{w}_0(\mathbf{x}) &= \mathbf{x} \\ \mathbf{w}_1(\mathbf{x}) &= \mathbf{A}\mathbf{x} + s_{N-1}\mathbf{b} \\ \mathbf{w}_2(\mathbf{x}) &= \mathbf{A}^2\mathbf{x} + s_{N-2}\mathbf{A}\mathbf{b} + s_{N-1}\mathbf{b} \\ &\vdots \\ \mathbf{w}_N(\mathbf{x}) &= \mathbf{A}^N\mathbf{x} + s_0\mathbf{A}^{N-1}\mathbf{b} + s_1\mathbf{A}^{N-2}\mathbf{b} + \dots + s_{N-2}\mathbf{A}\mathbf{b} + s_{N-1}\mathbf{b}, \end{aligned} \quad (3.6)$$

or via the general formula

$$\mathbf{w}_r(\mathbf{x}) = \mathbf{A}^r\mathbf{x} + \sum_{i=0}^{r-1} s_{N-1-i}\mathbf{A}^i\mathbf{b} \quad (3.7)$$

for $r = 0, 1, \dots, N$. (Note that we have employed the notation w_r for these functions in place of f_r which appears in their paper; this is to avoid confusion with the important functions f_r appearing in the Chua-Lin inequalities.) In Proposition 3.2, that follows, we deal with this result and show also its converse.

Proposition 3.2 (Extension of Theorem 1 in Galias & Ogorzałek 1992)

The string $s_0s_1\dots s_{N-1}$ is a substring of some admissible sequence for the map F if and only if the intersection $S = w_0(I^2) \cap w_1(I^2) \cap \dots \cap w_N(I^2)$ is non-empty.

Proof

If the string $s_0s_1\dots s_{N-1}$ appears as a substring of some admissible sequence then there is an admissible sequence which *commences* with the digits $s_0s_1\dots s_{N-1}$. (The justification is that if an admissible sequence contains the string $s_0s_1\dots s_{N-1}$ positioned so that it starts at the k^{th} digit, and \mathbf{x} is a point giving rise to that sequence, then the point $F^k(\mathbf{x})$ gives rise to an admissible sequence commencing $s_0s_1\dots s_{N-1}$). Let \mathbf{x} be a point in I^2 which generates an admissible sequence commencing $s_0s_1\dots s_{N-1}$, then

$$\begin{aligned} F(\mathbf{x}) &= A\mathbf{x} + s_0\mathbf{b} \\ F^2(\mathbf{x}) &= AF(\mathbf{x}) + s_1\mathbf{b} \\ &\vdots \\ F^N(\mathbf{x}) &= AF^{N-1}(\mathbf{x}) + s_{N-1}\mathbf{b} \end{aligned} \tag{3.8}$$

and all of the points $F(\mathbf{x})$, $F^2(\mathbf{x})$, \dots , $F^N(\mathbf{x})$ lie in the region I^2 . Working backwards through this system of equations, we can express the point $F^N(\mathbf{x})$ in terms of the functions w_1, w_2, \dots, w_N via

$$\begin{aligned} F^N(\mathbf{x}) &= AF^{N-1}(\mathbf{x}) + s_{N-1}\mathbf{b} \\ &= w_1(F^{N-1}(\mathbf{x})) = w_2(F^{N-2}(\mathbf{x})) \\ &= \dots \\ &= w_{N-1}(F(\mathbf{x})) = w_N(\mathbf{x}). \end{aligned} \tag{3.9}$$

It follows that each of the sets $w_1(I^2), w_2(I^2), \dots, w_N(I^2)$ contains the point $F^N(\mathbf{x})$. Finally, since $w_0(I^2) = I^2$, and $F^N(\mathbf{x}) \in I^2$, the intersection $S = w_0(I^2) \cap w_1(I^2) \cap \dots \cap w_N(I^2)$ contains the point $F^N(\mathbf{x})$, and hence is non-empty.

Conversely, suppose that the intersection $S = w_0(I^2) \cap w_1(I^2) \cap \dots \cap w_N(I^2)$ is non-empty and let \mathbf{x} be a point in this intersection. For each $r = 0, 1, \dots, N$, $\mathbf{x} \in w_r(I^2)$, so there corresponds a point in I^2 whose image under w_r is \mathbf{x} . And because w_r is invertible (as an affine map of the plane), there is only one such point, for each r . Denote these $(N+1)$ points as $\mathbf{x}_0, \mathbf{x}_1, \dots, \mathbf{x}_N$, where $w_r(\mathbf{x}_{N-r}) = \mathbf{x}$. We show that the point \mathbf{x}_0 gives rise to a sequence (necessarily admissible) commencing with the string $s_0 s_1 \dots s_{N-1}$.

For each $r = 0, 1, \dots, N-1$, $w_{N-(r+1)}(\mathbf{x}_{r+1}) = \mathbf{x} = w_{N-r}(\mathbf{x}_r)$ which may be written as

$$A^{N-r-1} \mathbf{x}_{r+1} + \sum_{i=0}^{N-r-2} s_{N-1-i} A^i \mathbf{b} = A^{N-r} \mathbf{x}_r + \sum_{i=0}^{N-r-1} s_{N-1-i} A^i \mathbf{b}, \quad (3.10)$$

which, after removing terms common to both summations, simplifies to

$$A^{N-r} \mathbf{x}_r + s_r A^{N-r-1} \mathbf{b} = A^{N-r-1} \mathbf{x}_{r+1}. \quad (3.11)$$

The matrix A is invertible so this becomes

$$\mathbf{x}_{r+1} = A \mathbf{x}_r + s_r \mathbf{b} \quad (3.12)$$

for $r = 0, 1, \dots, N-1$, which is precisely $\mathbf{x}_{r+1} = F(\mathbf{x}_r)$ because both \mathbf{x}_r and $\mathbf{x}_{r+1} \in I^2$. Thus under iteration of the map F , the point \mathbf{x}_0 generates a sequence which commences with the string $s_0 s_1 \dots s_{N-1}$. \square

The usefulness of Proposition 3.2 is that it provides an alternative means of establishing when a string $s_0 s_1 \dots s_{N-1}$ appears in some admissible sequence, *without having explicitly to construct a sequence containing the string*. This allows us to work with the strings themselves, to explore which strings form parts of admissible sequences and which do not, in a setting removed from the actual sequences.

3.3 A geometric reformulation of the criterion

The set $S = w_0(I^2) \cap w_1(I^2) \cap \dots \cap w_N(I^2)$ may be viewed as the intersection of $(N+1)$ geometric objects, thereby converting to a geometric problem the task of determining when a string $s_0 s_1 \dots s_{N-1}$ appears in some admissible sequence. From the geometric viewpoint, the sets $w_r(I^2)$ are simple: each is the region enclosed by a parallelogram. This is because the maps w_r are affine and area-preserving (since $\det(A) = 1$), and because the set I^2 is a (solid) square. However there is here one complication due to the fact (cf. our discussion in Section 2.1.4) that the set I^2 does not include the upper or right-hand boundaries of the square, and correspondingly each parallelogram $w_r(I^2)$ lacks two of its edges.

Our geometrical investigation for the strings $+0 \dots 0+$ would be unnecessarily complicated if we had constantly to take into account which boundaries of each of the parallelograms were missing, so instead our approach is to work entirely with closed sets, seeking the solution to the related geometric problem of determining precisely when the intersection

$$S' = w_0(\bar{I}^2) \cap w_1(\bar{I}^2) \cap \dots \cap w_N(\bar{I}^2), \quad (3.13)$$

is empty. Of course, we cannot then apply Proposition 3.2 directly, as the sets S' and S differ, but as we shall see, once the form of S' is known, it is usually a straightforward task to decide whether S is empty or not, and hence whether the string $s_0 s_1 \dots s_{N-1}$ appears in some admissible sequence for the map F . Thus for the rest of this section, and for Sections 3.4 to 3.8 which follow, our concern is the task of determining the set S' for some string $+0 \dots 0+$; we will return to the issue of the missing boundaries at the end of Section 3.8, where we use our understanding of S' for the strings $+0 \dots 0+$ to prove Proposition 3.1.

The set S' arising from a string $s_0 s_1 \dots s_{N-1}$ as in (3.13) is the intersection of closed parallelograms and, since intersection preserves convexity, it follows that S' is either empty or is a closed convex polygon. By way of example, Figure 3.1 is a computer plot of the sets $w_r(\bar{I}^2)$ for the string $+0-+$ when the parameter value $a = 1.5$, and shows a square (corresponding to the boundary of \bar{I}^2) along with four other parallelograms. Clearly in the

plot there is a region common to all five parallelograms (irrespective of whether boundaries are missing or included) and so by Proposition 3.2 the string $+0-+$ does appear in some admissible sequence when $a = 1.5$.

The problem of determining the set S' and its dependence on the parameter a is simplified by making the coordinate transformation introduced in Section 2.1.2. Letting $\mathbf{x} = \mathbf{T}\mathbf{y}$, where \mathbf{T} is the matrix $\begin{pmatrix} 1 & 0 \\ \cos\theta & \sin\theta \end{pmatrix}$ gives

$$\begin{aligned} \mathbf{w}_r(\mathbf{x}) &= \mathbf{T}\mathbf{R}_{r\theta}\mathbf{T}^{-1}\mathbf{x} + \sum_{i=0}^{r-1} s_{N-1-i}\mathbf{T}\mathbf{R}_{i\theta}\mathbf{T}^{-1}\mathbf{b} \\ &= \mathbf{T}\left(\mathbf{R}_{r\theta}\mathbf{y} + \sum_{i=0}^{r-1} s_{N-1-i}\mathbf{R}_{i\theta}\mathbf{T}^{-1}\mathbf{b}\right) \end{aligned} \quad (3.14)$$

so the sets S and S' correspond respectively (via the coordinate change) to the sets

$$\bar{S} = \bar{\mathbf{w}}_0(\mathbf{T}^{-1}(I^2)) \cap \bar{\mathbf{w}}_1(\mathbf{T}^{-1}(I^2)) \cap \dots \cap \bar{\mathbf{w}}_N(\mathbf{T}^{-1}(I^2)) \quad (3.15)$$

and

$$\bar{S}' = \bar{\mathbf{w}}_0(\mathbf{T}^{-1}(I'^2)) \cap \bar{\mathbf{w}}_1(\mathbf{T}^{-1}(I'^2)) \cap \dots \cap \bar{\mathbf{w}}_N(\mathbf{T}^{-1}(I'^2)) \quad (3.16)$$

where

$$\bar{\mathbf{w}}_r(\mathbf{y}) = \mathbf{R}_{r\theta}\mathbf{y} + \sum_{i=0}^{r-1} s_{N-1-i}\mathbf{R}_{i\theta}\mathbf{T}^{-1}\mathbf{b}. \quad (3.17)$$

The transformation \mathbf{T} is non-singular so the set \bar{S} is empty precisely when S is; hence Proposition 3.2 can be equivalently stated in terms of whether \bar{S} is empty or not, and our comments above about the relation between S and S' apply equally to \bar{S} and \bar{S}' .

The change of coordinates is advantageous because each of the functions $\bar{\mathbf{w}}_r$ is simply the composition of a rigid rotation and a translation. So whereas the $(N+1)$ parallelograms forming the intersection S' in the original coordinate system all have different shapes (as is clearly to be seen in Figure 3.1), in the new coordinate system the intersection \bar{S}' is formed with $(N+1)$ *congruent* geometric objects, varying only in amount of rotation and

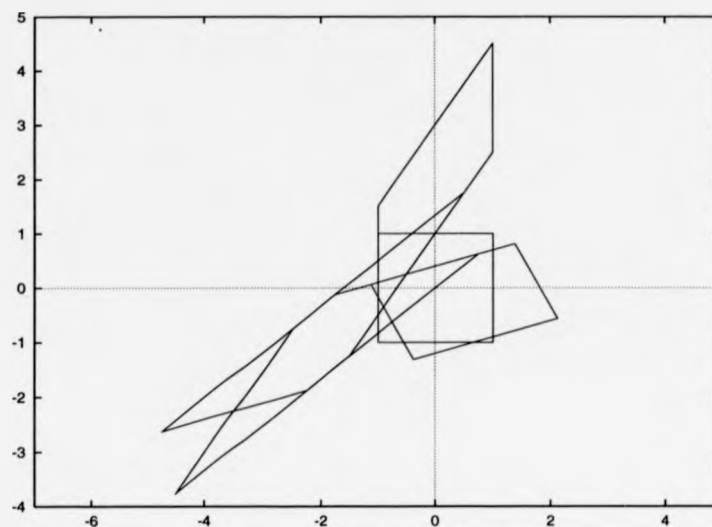


Figure 3.1: The sets $w_0(I^2)$, $w_1(I^2)$, \dots , $w_4(I^2)$ for the string $+0-+$ at parameter value $a = 1.5$. Each set is a (solid) parallelogram.

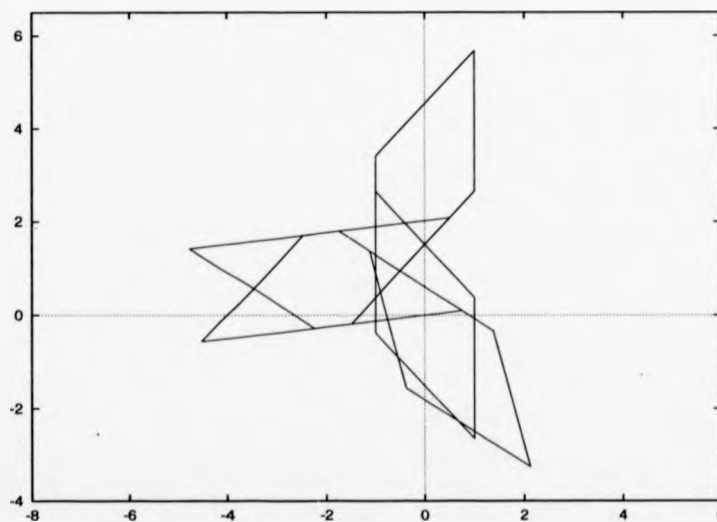


Figure 3.2: The plot of Figure 3.1 after a change of coordinates by the linear map T . In the new coordinates the intersection arises from the congruent (solid) rhombi $\tilde{w}_0(T^{-1}(I^2))$, $\tilde{w}_1(T^{-1}(I^2))$, \dots , $\tilde{w}_4(T^{-1}(I^2))$.

in distance of translation from the origin. Figure 3.2 is a plot for the same string $+0-+$, but in the new coordinates; here the intersection is between five identically shaped objects.

It is essential in the work which follows to have a good understanding of the set $\mathbf{T}^{-1}(\bar{I}^2)$, as this set is the basic component from which the intersection \bar{S}^g arises. In the remainder of this section we present a thorough description of the geometry of the set $\mathbf{T}^{-1}(\bar{I}^2)$, and we pay particular attention to how this set evolves as the parameter θ is increased. This is because our overall aim is to find the parameter ranges for which the string $+0 \dots 0+$ appears in some admissible sequence, and the parameter θ determines not only the *locations* of the copies of $\mathbf{T}^{-1}(\bar{I}^2)$ whose common intersection is \bar{S}^g , but also the *shape* of the set $\mathbf{T}^{-1}(\bar{I}^2)$ itself.

The geometry of the set $\mathcal{R} = \mathbf{T}^{-1}(\bar{I}^2)$

Direct calculation shows that the coordinates of the vertices of $\mathbf{T}^{-1}(\bar{I}^2)$ are :

$$\begin{aligned} a &: \left(-1, \frac{\cos \theta - 1}{\sin \theta}\right) & b &: \left(1, \frac{-\cos \theta - 1}{\sin \theta}\right) \\ c &: \left(1, \frac{1 - \cos \theta}{\sin \theta}\right) & d &: \left(-1, \frac{1 + \cos \theta}{\sin \theta}\right). \end{aligned}$$

Two of the edges, ad and bc , are vertical and of length $\frac{2}{\sin \theta}$. The lengths of the other two edges are readily calculated as

$$|ab| = |cd| = \left\{ 2^2 + \left(\frac{2\cos \theta}{\sin \theta} \right)^2 \right\}^{\frac{1}{2}} = \frac{2}{\sin \theta},$$

so all four edge lengths are equal and the set $\mathbf{T}^{-1}(\bar{I}^2)$ is a (solid) rhombus. For notational convenience, we will henceforth denote the rhombus $\mathbf{T}^{-1}(\bar{I}^2)$ by \mathcal{R} .

There are several ways to find the internal angles of the rhombus, although some care is needed because these angles change from acute to obtuse, and vice-versa, as the parameter θ passes through $\frac{\pi}{2}$. A method applicable for all $0 < \theta < \pi$ is to include the diagonal ac to create an isosceles triangle $\triangle acd$. The length of ac is determined from the coordinates above as $\frac{2\sqrt{2-2\cos \theta}}{\sin \theta}$, which means that the internal angle at the vertex d is simply θ .

Figure 3.3 displays the rhombus \mathcal{R} , with vertices and angles indicated, for three different and typical parameter values θ . Notice how the shape of the rhombus varies with θ . The edge lengths $\frac{2}{\sin \theta}$ increase indefinitely as θ approaches 0 or π . When θ is close to zero, \mathcal{R} is long and thin and bd is the longer of the two diagonals. As θ approaches $\frac{\pi}{2}$ the inequality of the diagonal lengths disappears, and at $\theta = \frac{\pi}{2}$ all the internal angles are right-angles, so that \mathcal{R} is a square. When θ exceeds $\frac{\pi}{2}$, the internal angles at b and d are obtuse and the diagonal ac is the longer diagonal. Finally the rhombus \mathcal{R} becomes long and thin once more as θ approaches π .

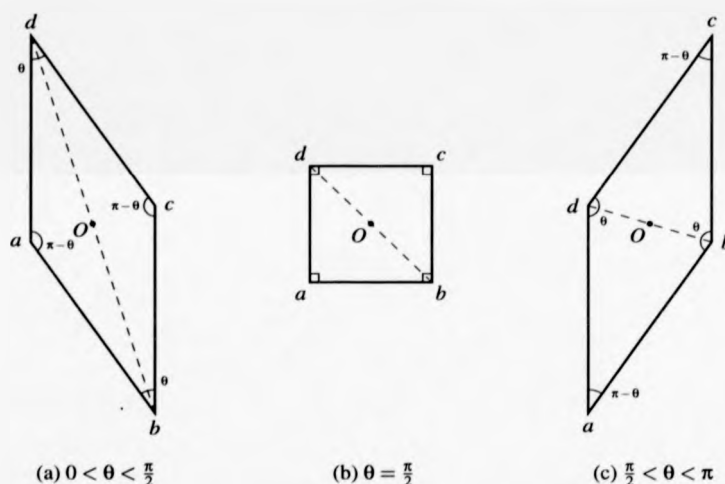


Figure 3.3: The geometry of the rhombus \mathcal{R} .

Because each copy of \mathcal{R} within the collection of rhombi $\{\bar{\mathbf{w}}_r(\mathcal{R}) : r = 0, 1, \dots, N\}$ is rotated and translated by a different amount, and the extent of each rotation and translation depends on θ , to keep track of the orientations of the rhombi it is useful to designate one vertex of \mathcal{R} as a reference. We choose vertex d , and label it the *prime vertex*; correspondingly the diagonal bd is labelled the *prime diagonal*; both have been marked on the plots of Figure 3.3. (Note that the “prime diagonal” is not synonymous with the “longer diagonal”: when $\theta > \frac{\pi}{2}$ it is the shorter of the two diagonals.) We will frequently specify the

orientation of a rotated copy of \mathcal{R} in terms of the angle made by its prime diagonal with the vertical; specifically we quote $\angle vOd$ between the prime vertex d and the upward vertical v , and it will prove most convenient to measure this angle in a clockwise sense, to accord with the clockwise rotations \mathbf{R}_θ . The orientation of the prime diagonal for the rhombus \mathcal{R} is $\angle vOd = -\frac{1}{2}\theta$, which is an easy consequence of the fact that the prime diagonal bisects the internal angle θ at d .

On occasions we will refer to the orientations of the individual edges of a rotated copy of rhombus \mathcal{R} . We measure these angles from a local upward vertical, again in a clockwise sense. For the rhombus \mathcal{R} itself, two of the edges are vertical, so have angle 0, and the two sloping edges are at an angle of $-\theta$ when measured clockwise from the local upward vertical.

Although the shape of \mathcal{R} varies with the parameter θ , some aspects of its geometry are independent of θ . These include :

- (i) The two diagonals meet at the origin O , so \mathcal{R} remains centred on the origin.
- (ii) The two edges ad and bc are always vertical.
- (iii) The perpendicular distance from the centre of \mathcal{R} to each of its edges is 1.

To justify the last of these points, note that the perpendicular distance to each of the two vertical edges is known from the horizontal coordinates of the vertices. The result then follows for the other two edges because the rhombus has reflection symmetry in the diagonal bd .

Point (iii) turns out to be a crucial observation, because it means that a unit circle centred at O can be inscribed in \mathcal{R} , tangent to the four edges of the rhombus. We call the four points where the edges of \mathcal{R} meet the circle p_1 to p_4 (these are the tangent points for the edges cd , bc , ab and ad respectively), and describe their locations on the circle by the angles they subtend at O , again measured clockwise from the upward vertical. Figure 3.4 shows the rhombus \mathcal{R} with the unit circle inscribed and the *points of contact* p_1 to p_4 marked. The points p_2 and p_4 , on the vertical edges of the rhombus, remain fixed as θ varies, and have

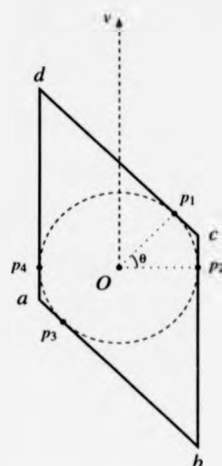


Figure 3.4: A unit circle may be inscribed in the rhombus \mathcal{R} , touching its four sides.

position angles $\angle vOp_2 = \frac{\pi}{2}$ and $\angle vOp_4 = \frac{3\pi}{2}$. The angle describing the location of p_1 is obtained by considering the quadrilateral Op_1cp_2 , from which we find $\angle p_1Op_2 = \theta$, so $\angle vOp_1 = \frac{\pi}{2} - \theta$. Finally p_3 is on the opposite edge to p_1 , so its angle leads that of p_1 by π , i.e. $\angle vOp_3 = \frac{3\pi}{2} - \theta$. Note that while Figure 3.4 is drawn for $\theta < \frac{\pi}{2}$, the argument given and the formulae for the angles of p_1 to p_4 also hold when θ is obtuse, but in this case the angle $\angle vOp_1$ is negative because p_1 lies to the left of the vertical axis.

The importance of this inscribed unit circle is that it can be used to give an alternative description of the rhombus. \mathcal{R} may be thought of as the intersection of four half-planes whose edges are the tangent lines to the unit circle at p_1 , p_2 , p_3 and p_4 , and this viewpoint is an advantageous one for many of the arguments in the subsequent sections.

We finish this section by listing, in Table 3.2, the lengths and angles which describe the geometry of the rhombus \mathcal{R} .

LengthsLength of edges : $\frac{2}{\sin \theta}$ Lengths of diagonals : $\frac{2}{\sin \frac{1}{2}\theta}$ (prime diagonal bd), $\frac{2}{\cos \frac{1}{2}\theta}$ (diagonal ac)**Angles**Internal angles : θ (angles b and d), $\pi - \theta$ (angles a and c)Inclination of prime diagonal : $\angle vOd = -\frac{1}{2}\theta$ Inclination of edges : 0 (edges ad and bc), $-\theta$ (edges ab and cd)
(from local vertical)Position angles of points of contact : $\angle vOp_1 = \frac{\pi}{2} - \theta$, $\angle vOp_2 = \frac{\pi}{2}$, $\angle vOp_3 = \frac{3\pi}{2} - \theta$,
 $\angle vOp_4 = \frac{5\pi}{2}$ Table 3.2: Lengths and angles associated with the rhombus \mathcal{R} .**3.4 The rhombus configuration for the strings $+0 \dots 0+$**

Our presentation in Sections 3.2 and 3.3 has been set out in terms of a general string $s_0 s_1 \dots s_{N-1}$. At this stage we will specialise to the particular family of strings under investigation, namely the strings $+0 \dots 0+$. Here we describe the configuration of the $(N+1)$ rhombi $\bar{w}_r(\mathcal{R})$ arising from the length N string $+0 \dots 0+$ whose intersection is \bar{S}' . The run of zeros present in the string $+0 \dots 0+$ means that expressions for these particular functions \bar{w}_r are simpler than the general forms stated above. Indeed it is this simplification which gives the configuration a regular form, permitting us to track the evolution of the intersection \bar{S}' as θ varies, so that ultimately we can determine precisely the parameter range for which the string $+0 \dots 0+$ appears in some admissible sequence.

When we set $s_0 = s_{N-1} = +$, $s_1 = s_2 = \dots = s_{N-2} = 0$, for the string $+0 \dots 0+$ of length N , in equation (3.17) for the \bar{w}_r , we obtain :

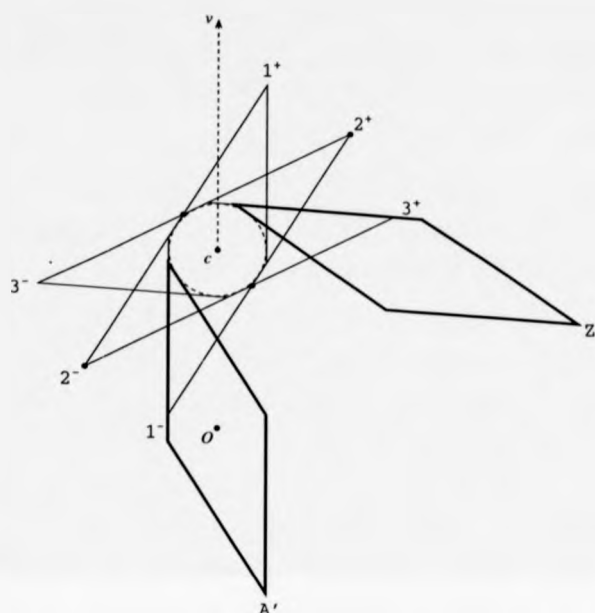
$$\begin{aligned}
\bar{w}_0(y) &= y \\
\bar{w}_1(y) &= R_\theta y + T^{-1}b \\
\bar{w}_2(y) &= R_{2\theta} y + T^{-1}b \\
&\vdots \\
\bar{w}_{N-1}(y) &= R_{(N-1)\theta} y + T^{-1}b \\
\bar{w}_N(y) &= R_{N\theta} y + R_{(N-1)\theta} T^{-1}b + T^{-1}b.
\end{aligned} \tag{3.18}$$

As was the case for a general string $s_0 s_1 \dots s_{N-1}$, each of these functions represents a rotation followed by a translation, so each of the rhombi $\bar{w}_r(\mathcal{R})$ is a copy of \mathcal{R} rotated about its centre by an amount $r\theta$ in a clockwise sense and then translated. The effect of the run of zeros in the string $+0 \dots 0+$ is that the translation components of the functions $\bar{w}_1, \bar{w}_2, \dots, \bar{w}_{N-1}$ coincide, they all are

$$T^{-1}b = \frac{1}{\sin\theta} \begin{pmatrix} \sin\theta & 0 \\ -\cos\theta & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{2}{\sin\theta} \end{pmatrix}, \tag{3.19}$$

i.e. a displacement by $\frac{2}{\sin\theta}$ in the upwards vertical direction. Thus the collection of rhombi $\bar{w}_0(\mathcal{R}), \bar{w}_1(\mathcal{R}), \dots, \bar{w}_N(\mathcal{R})$, whose intersection \bar{S}' we wish to find, can be split into two groups, namely :

- (i) $(N-1)$ *central rhombi* which we denote by $1, 2, \dots, (N-1)$, and which are the sets $\bar{w}_1(\mathcal{R}), \bar{w}_2(\mathcal{R}), \dots, \bar{w}_{N-1}(\mathcal{R})$ respectively. Central rhombus r is obtained from \mathcal{R} by a rotation of $r\theta$ clockwise about its centre, followed by a translation of $\frac{2}{\sin\theta}$ vertically upwards. All the central rhombi have their centres at the same point, $c = (0, \frac{2}{\sin\theta})$, and each is inclined at an angle θ to its predecessor.
- (ii) 2 *eccentric rhombi* which we denote by A' and Z' . Rhombus A' is $\bar{w}_0(\mathcal{R})$, which is simply \mathcal{R} , and rhombus Z' is $\bar{w}_N(\mathcal{R})$. The location and orientation of eccentric rhombus Z' is most simply stated relative to the final central rhombus; Z' is obtained from central rhombus $(N-1)$ by a further rotation of θ about its centre followed by an additional translation away from c , of a distance $\frac{2}{\sin\theta}$ in the direction making an angle $(N-1)\theta$ measured clockwise from the upward vertical.

Figure 3.5: The rhombus configuration for the string $+00+$.

The $(N + 1)$ rhombi $A', 1, 2, \dots, (N-1), Z'$ collectively make up the *rhombus configuration* for the string $+0 \dots 0+$. Figure 3.5 is an illustration for the case $N = 4$, and shows the rhombus configuration for the string $+00+$. We adopt the convention of labelling the two ends of the prime diagonal for each of the central rhombi: a $+$ superscript indicates the prime vertex, and a $-$ superscript is used at the other end of the diagonal. This facilitates the identification of the orientations of the central rhombi, which can otherwise become difficult at larger values of θ ; we will see examples of this in the next section. Mostly it is not necessary to distinguish the prime vertices of the eccentric rhombi. (As a practical matter, to aid the recognition of the separate rhombi in the diagrams of rhombus configurations for the string $+00+$, we have picked out the vertices of central rhombus 2.)

The description of the eccentric rhombi in paragraph (ii) above was motivated by their definitions from the formulae (3.18), but this is not the most natural description for our

study of the rhombus configuration and we now give an alternative which makes clearer the locations of A' and Z' in relation to the central rhombi. If we write the formulae for the functions \bar{w}_0 and \bar{w}_N from (3.18) as

$$\bar{w}_0(y) = (y + T^{-1}b) - T^{-1}b$$

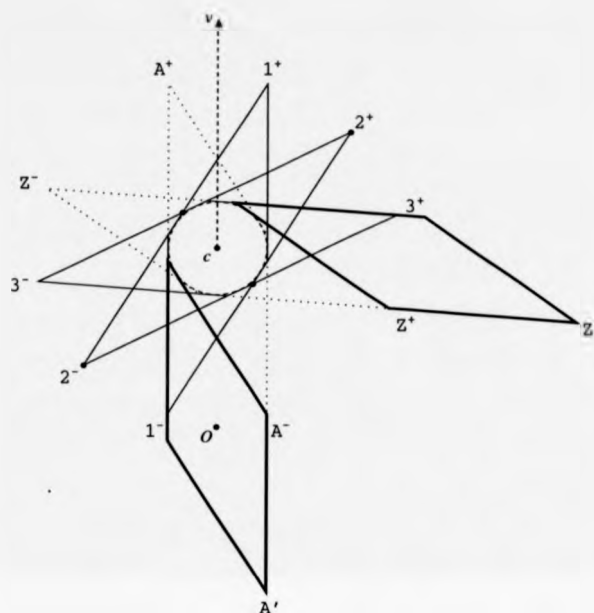
and

$$\bar{w}_N(y) = (R_{N\theta}y + T^{-1}b) + R_{(N-1)\theta}T^{-1}b,$$

we see that if the eccentric rhombi A' and Z' were translated by $T^{-1}b$ and $-R_{(N-1)\theta}T^{-1}b$ respectively we would obtain two new central rhombi. We will call these extra rhombi A and Z . Rhombus A is the predecessor of central rhombus 1 (because rotating A by θ clockwise about c maps it onto 1), and similarly rhombus Z is the successor of central rhombus $(N-1)$.

Of course, A and Z do not belong to the rhombus configuration; we introduce them solely as an aid to describe the locations of A' and Z' , each of which may now be viewed as a simple translation of a central rhombus. Since the translation taking A to A' is in the vertical direction, parallel to two of the edges of A , and is by a distance exactly equal to the length of an edge, $\frac{2}{\sin\theta}$, rhombi A and A' have an edge in common, namely the lower edge of A and the upper edge of A' . Likewise, the translation from Z to Z' is of length $\frac{2}{\sin\theta}$ in a direction making an angle $(N-1)\theta$ with the upward vertical, and this direction is parallel to two of the edges of Z' , so Z and Z' also have an edge in common. (To see this, note that the orientations of the edges of Z' are obtained by rotating the edges of \mathcal{R} by an angle $N\theta$ clockwise. The edges of \mathcal{R} may be described as making angles $-\theta$ and 0 from the upward vertical, so those of Z' are at angles $(N-1)\theta$ and $N\theta$.)

Figure 3.6 is a second plot of the configuration depicted in Figure 3.5, and here the extra central rhombi A and Z have been included, so the plot exhibits how A' and Z' may be viewed as arising from translations of these new central rhombi. An easy way to visualise these translations parallel to the sides of A and Z is that each is the first step of a tiling of a strip, for the case of A and A' a vertical strip, and for Z and Z' a strip aligned at an angle $(N-1)\theta$.

Figure 3.6: The configuration for $+00+$ with central rhombi A and Z added.

There are other ways to describe the locations of the rhombi A' , 1 , 2 , \dots , $(N-1)$, Z' for a configuration specified by N and θ . Two of these are set out below, and relate to features of the rhombus \mathcal{R} introduced in Section 3.3. The first specifies the positions and orientations of each of the prime diagonals, and the second gives the locations of the points of contact of the various rhombi with the central circle.

(a) **Specification of the rhombus configuration by the prime diagonals**

The location and orientation of each rhombus may be fully specified by giving the centre point together with the angle of inclination of its prime diagonal. All central rhombi have centre point $c = (0, \frac{2}{\sin \theta})$. We saw in the last section that the angle of inclination for the original rhombus \mathcal{R} was $-\frac{1}{2}\theta$. The orientations of the central rhombi arise from successive clockwise rotations by the angle θ from the rhombus \mathcal{R} ; thus the angles of inclination of their prime diagonals are $\angle vc1^+ = \frac{1}{2}\theta$, $\angle vc2^+ = \frac{3}{2}\theta$, \dots , $\angle vc(N-1)^+ = (N - \frac{3}{2})\theta$.

The eccentric rhombi arise through the translations of A and Z as described above. The orientations of their prime diagonals are $-\frac{1}{2}\theta$ for A' and $(N - \frac{1}{2})\theta$ for Z' (these angles being measured at the centre of each rhombus, clockwise from the *local* upward vertical).

(b) **Specification of the rhombus configuration by the points of contact**

Because all $(N - 1)$ central rhombi have the same centre point c , the four edges of each central rhombus are tangent to a single unit circle centred at c . We will call this circle the *central circle*. Each central rhombus has four points of contact on this circle, the angles describing their locations on the circle are obtained from the angles of the points of contact of \mathcal{R} , simply by adding the appropriate multiple of θ , i.e. for the r^{th} central rhombus, arising from a clockwise rotation by $r\theta$ of \mathcal{R} , we add $r\theta$ to each of the angles of the points of contact for \mathcal{R} . The following table lists the angles of the points of contact for the central rhombi.

| Central Rhombus | Position Angles of Points of Contact | | | |
|-----------------|--------------------------------------|-------------------------------|--------------------------------|--------------------------------|
| | p_1 | p_2 | p_3 | p_4 |
| 1 | $\frac{\pi}{2}$ | $\frac{\pi}{2} + \theta$ | $\frac{3\pi}{2}$ | $\frac{3\pi}{2} + \theta$ |
| 2 | $\frac{\pi}{2} + \theta$ | $\frac{\pi}{2} + 2\theta$ | $\frac{3\pi}{2} + \theta$ | $\frac{3\pi}{2} + 2\theta$ |
| \vdots | \vdots | \vdots | \vdots | \vdots |
| (N-1) | $\frac{\pi}{2} + (N-2)\theta$ | $\frac{\pi}{2} + (N-1)\theta$ | $\frac{3\pi}{2} + (N-2)\theta$ | $\frac{3\pi}{2} + (N-1)\theta$ |

The eccentric rhombi have only a single point of contact with the central circle. This occurs on the edge in common with rhombus A or Z, which makes it straightforward to write down the angle for the position of the point of contact on the central circle; the operative points of contact are p_3 on A and p_4 on Z, so the angles are :

| Eccentric Rhombus | Position Angle of Point of Contact |
|-------------------|------------------------------------|
| A' | $\frac{3\pi}{2} - \theta$ |
| Z' | $\frac{3\pi}{2} + N\theta$ |

These alternative descriptions both have their merits, and each will be employed in the course of our reasoning to establish the values of θ for which the common intersection of the rhombi in the configuration is non-empty. However in many instances the viewpoint (b), whereby a central rhombus is regarded as the intersection of four half planes with edges the tangents at the four points of contact, lends extra insight and leads to sharper proofs. To illustrate this, we give here a simple example, proving an elementary result about the rhombus configuration.

Example 3.3 : Successive central rhombi share two edges

Reference to the configuration drawn in Figure 3.5 reveals that successive central rhombi share parts of the length of two of their sides. This is not a special feature of the configuration in Figure 3.5; indeed it is true of any rhombus configuration for the string $+0 \dots 0+$. The result follows because successive central rhombi have two of their points of contact in common (see the table in paragraph (b) above).

Clearly this is not a deep or difficult result, but the ease with which it is established using the points of contact contrasts with the effort which would be required to calculate the locations of the individual edges either by coordinates or from the angles of orientation of the prime diagonals.

3.5 Evolution of the rhombus configuration as θ increases

Before we commence the theory describing the precise evolution of the general rhombus configuration with increasing parameter θ , it will be helpful to investigate carefully an individual case. So in this section we select one specific short length string and study the effect of varying θ on its rhombus configuration. The approach we adopt is a graphical one : Figure 3.7, on pages 78 to 81 exhibits sixteen plots of the rhombus configuration for the length $N = 4$ string $+00+$. The plots show the evolution of the 3 central rhombi and 2 eccentric rhombi for this string as θ increases through its complete range from 0 to π . The sixteen values of θ chosen for the plots have been selected to illustrate the important stages

in the evolution of the rhombus configuration.

The extended example presented here serves two purposes. Firstly, the evolution of the rhombus configuration for the string $+00+$ is not untypical of that for a general string $+0 \dots 0+$, illustrating many of the features which occur in the configurations corresponding to longer strings. And secondly, this particular example will illuminate and motivate the general theory when we encounter it later.

If we refer to the statement of Proposition 3.1 and ascertain from it the ranges of θ for which the string $+00+$ appears in some admissible sequence, we find the two intervals $(\frac{\pi}{3}, \frac{\pi}{2})$ and $(\frac{3\pi}{4}, \pi)$, so accordingly we are to expect to find a common intersection for the five rhombi A' , 1, 2, 3, Z' throughout these two intervals. The paragraphs below discuss how these two regimes of common intersection arise from the evolution of this particular rhombus configuration, referring in turn to each of the plots in Figure 3.7.

We should mention here that several of the plots have had to be reduced in size to fit on a page. The amount of the reduction applied to each may be inferred from the plotted size of the central circle (since the radius of the central circle is independent of θ and remains fixed at 1, so in the absence of scaling it would be drawn identically in each plot). For the plots near $\theta = 0$ or $\theta = \pi$ where the rhombi are greatly elongated, a greater reduction in size was needed than for the plots near $\theta = \frac{\pi}{2}$.

Plot (a) : The configuration when θ is close to zero

For small values of the parameter θ , the rhombi have much greater length than width. There is no common intersection because the two eccentric rhombi are well separated : rhombus A' lies wholly below the horizontal line through the centre of the central circle, whilst rhombus Z' lies wholly above this line.

Plots (b)-(c) : Eccentric rhombus Z' moves away from its starting location

As θ begins to increase, the three central rhombi, which were initially close, start to separate (compare plots (a), (b) and (c) to see this). The reason for the separation is that rhombi 2 and 3 are rotated copies of rhombus 1 by θ and 2θ respectively, so the rate of rotation with

respect to θ of rhombus 3 is twice that of rhombus 2. As the point of contact of eccentric rhombus Z' with the central circle starts to move clockwise around the central circle, the rhombus itself swings round and begins to approach eccentric rhombus A' , whose centre is fixed at the origin.

Plot (d) : The eccentric rhombi first overlap

Shortly before $\theta = 0.85$, eccentric rhombus Z' first meets A' . As θ continues to increase a region of overlap of Z' and A' appears and begins to grow in size.

Plots (e)-(f) : A common intersection for rhombi A' , 1, 2, 3, Z'

The two plots (e) and (f) are either side of the critical value $\theta = \frac{\pi}{3}$ at which a common intersection of A' , 1, 2, 3, Z' first occurs. In plot (f) a small region belonging to all five rhombi can be seen, located below the central circle.

Plots (g)-(h) : Evolution of the common intersection

As θ increases beyond $\frac{\pi}{3}$, the region where the five rhombi intersect initially begins to grow, but then recedes as θ approaches $\frac{\pi}{2}$.

Plot (i) : The common intersection ends

Plot (i) is for $\theta = \frac{\pi}{2}$, and when we reach this value the region of intersection has shrunk to a single point at the coincident corners of A' and Z' . The rather confusing appearance of the configuration in plot (i) is caused by the three central rhombi being instantaneously coincident at $\theta = \frac{\pi}{2}$, and is compounded because the two diagonals of each rhombus now have the same length, which makes it difficult to keep track of their orientations.

Plot (j) : Central rhombus 3 overtakes rhombus 1

Eccentric rhombi A' and Z' cease to overlap as θ increases beyond $\frac{\pi}{2}$, so the region of intersection disappears. As the three central rhombi begin to separate from their coincident alignment at $\frac{\pi}{2}$, the faster moving rhombus 3 has caught up with the "tail" of rhombus 1, so the prime vertex 3^+ of rhombus 3 emerges ahead of the vertex 1^- of rhombus 1. As θ passes $\frac{\pi}{2}$ the order of the vertices reading clockwise around the diagram changes from $1^+, 2^+, 3^+, 1^-, 2^-, 3^-$, as it has been throughout plots (a) to (h), to $1^+, 3^-, 2^+, 1^-, 3^+, 2^-$.

Plots (k)-(l) : A range of θ with no common intersection

As θ continues to increase beyond $\frac{\pi}{2}$ the eccentric rhombi A' and Z' become well separated, with the whole extent of the central circle lying between them while Z' swings around the top of the central circle. Eventually, as θ nears $\frac{3\pi}{4}$, one of the vertices of Z' starts to approach the upper right vertex of A' . Note that in plots (k), (l) and all subsequent ones, because θ is an obtuse angle and consequently the prime diagonal of each rhombus is now the shorter of the two diagonals, the labels at the ends of the prime diagonals of the central rhombi begin to cluster around the central circle, an effect which becomes more and more marked as θ becomes larger.

Plots (m)-(n) : A second range of θ where $A', 1, 2, 3, Z'$ intersect

At $\theta = \frac{3\pi}{4}$, in plot (m), eccentric rhombi A' and Z' meet for a second time and share a vertex. Because this vertex is also at the intersection of sides of rhombi 1 and 3 and within rhombus 2, all of $A', 1, 2, 3$ and Z' have a common intersection again. As θ increases (plot (n)) a region of common intersection emerges, and there is a second range for θ where all of $A', 1, 2, 3, Z'$ intersect.

Plots (o)-(p) : The common intersection persists as θ approaches π

As θ approaches π , the rhombi become long and thin again, but unlike the configuration near $\theta = 0$, there is a common intersection which persists as θ approaches π .

The descriptions of the plots have been written so as to give an intuitive feel for the evolution of the configuration, and vague terms such as "swing round" or "overtake" have been used intentionally in order to avoid introducing all the technical details of the process of evolution. In Section 3.7 we will make precise all of these ideas, and examine how much of what has been discussed here carries over to strings of arbitrary length. One point which should be stressed, though, is that the evolution of the rhombus configuration is *not* a purely rotational phenomenon : it is complicated greatly because, as well as the orientations, the shape and locations of the rhombi alter with θ . If the effect was purely rotational, determining when a common intersection was present would be a very much easier problem. However, in Section 3.7, we shall see that ultimately our method of analysis is to isolate the rotational aspect of the rhombus configuration evolution.

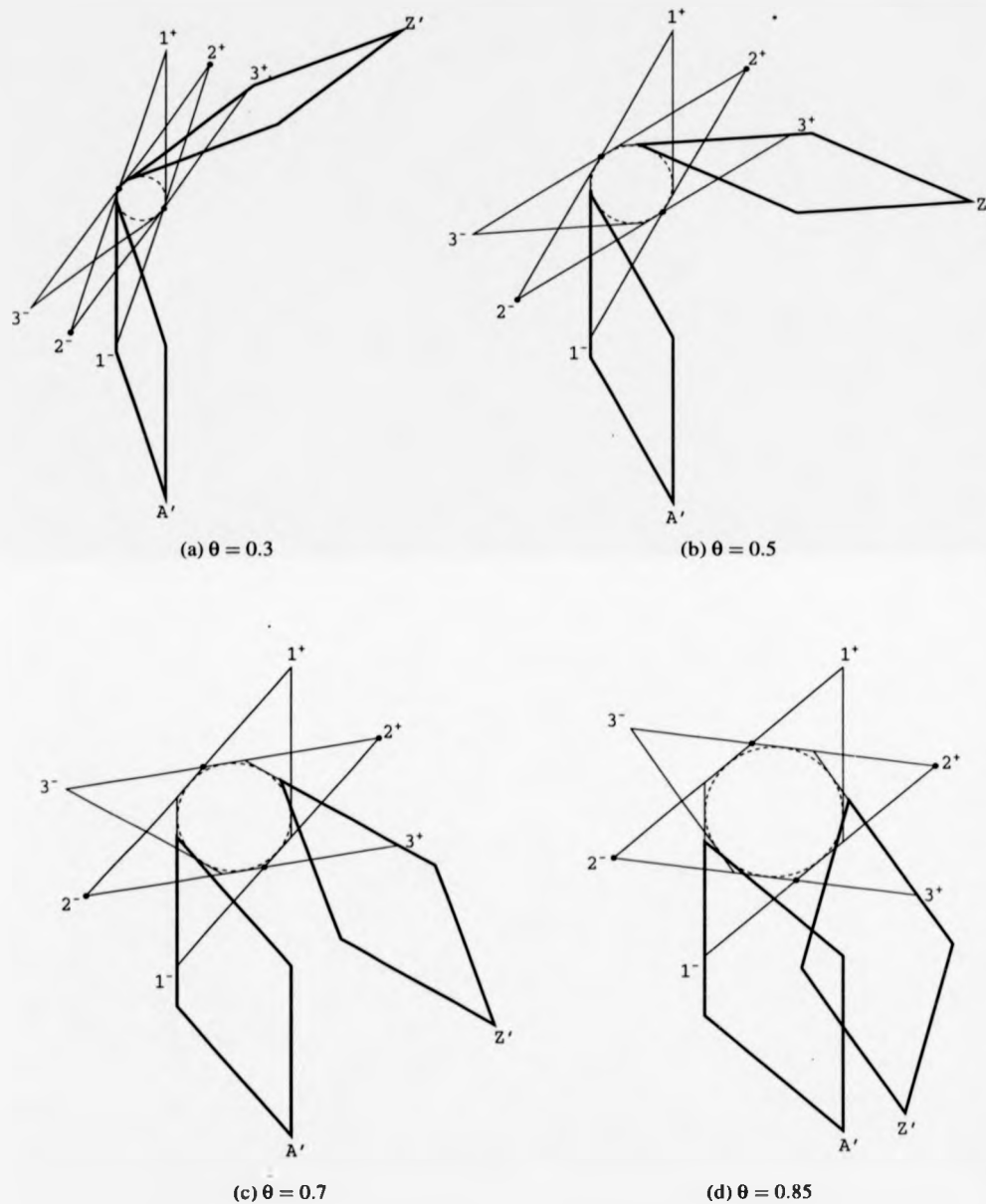


Figure 3.7: Plots of the rhombus configuration for $+00+$, illustrating its evolution with increasing θ .

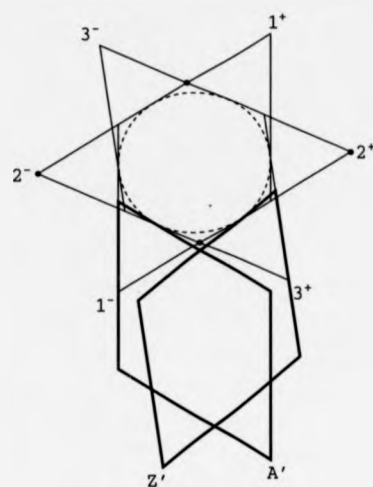
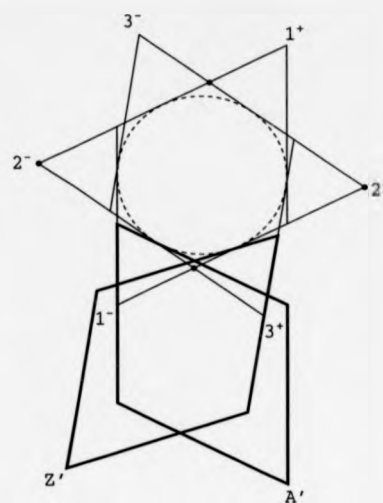
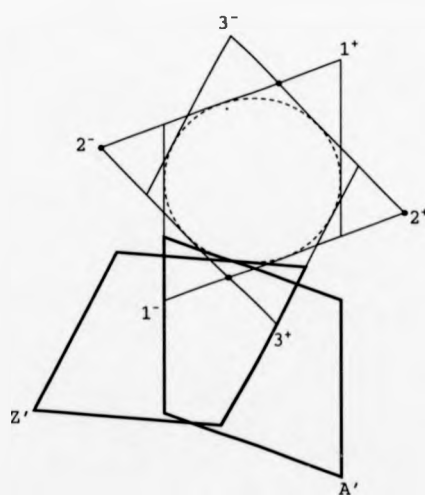
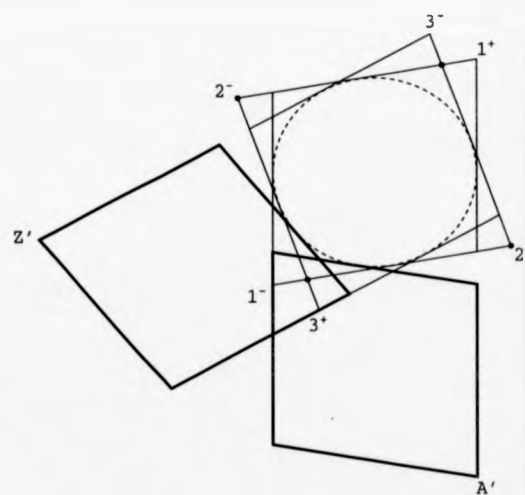
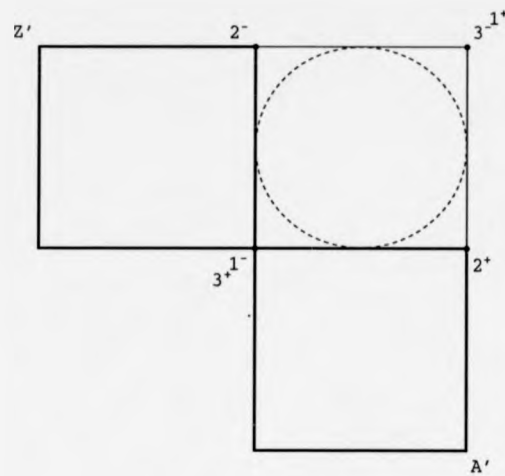
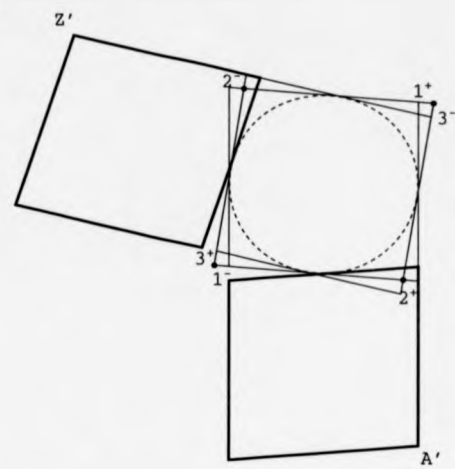
(e) $\theta = 1$ (f) $\theta = 1.1$ (g) $\theta = 1.2$ (h) $\theta = 1.4$

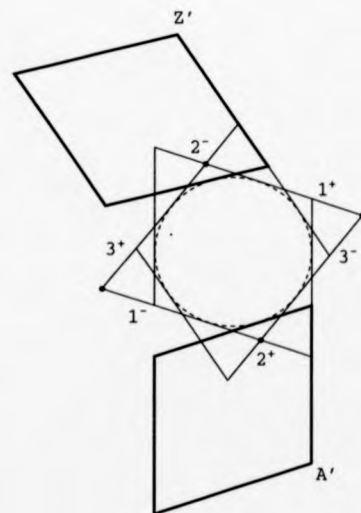
Figure 3.7: continued.



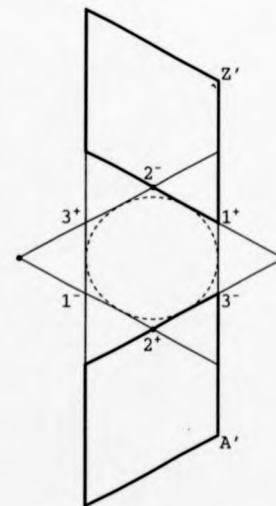
(i) $\theta = 1.570796327 \approx \frac{\pi}{2}$



(j) $\theta = 1.65$

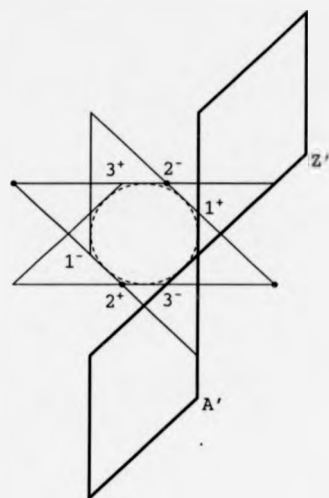


(k) $\theta = 1.9$

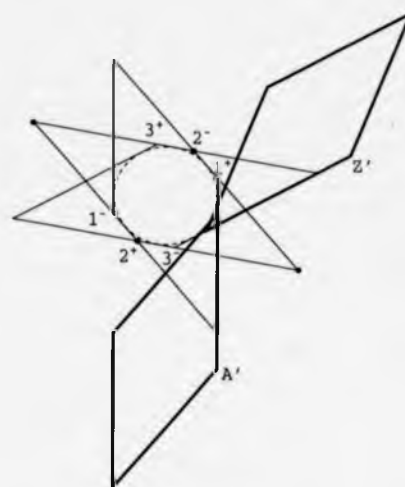


(l) $\theta = 2.094395102 \approx \frac{2\pi}{3}$

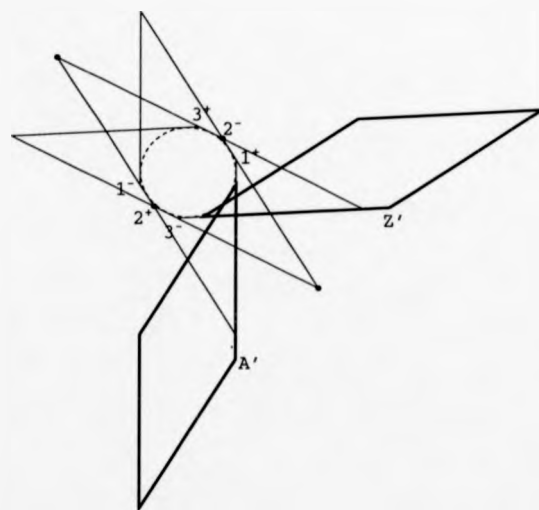
Figure 3.7: continued.



(m) $\theta = 2.35619449 \approx \frac{3\pi}{4}$



(n) $\theta = 2.45$



(o) $\theta = 2.6$



(p) $\theta = 2.9$

Figure 3.7: continued.

3.6 Symmetries of the rhombus configuration

In this section we investigate some symmetry properties of the rhombus configuration for the string $+0 \dots 0+$. These will be central to our argument in Section 3.7, where we determine the range of θ for there to be a common intersection of all the rhombi. The first of the symmetries is a reflection in an axis through the point c , the centre of the central circle.

Proposition 3.3

The rhombus configuration for the string $+0 \dots 0+$ of length N is symmetric under reflection in an axis σ passing through c and inclined at an angle $\frac{1}{2}(N-1)\theta + \frac{\pi}{2}$ to the upward vertical.

Proof

We start by demonstrating the symmetry for the central rhombi $1, 2, \dots, (N-1)$. We consider the prime diagonals of the central rhombi and define σ as the axis passing through c which bisects $\angle 1^+c(N-1)^-$, so that the prime diagonals of rhombi 1 and $(N-1)$ are conjugate under reflection in σ . Since each prime diagonal arises from its predecessor by a clockwise rotation of θ about c , the axis σ also bisects each of the angles $\angle 2^+c(N-2)^-$, $\angle 3^+c(N-3)^-$, ..., $\angle (N-1)^+c1^-$ (see Figure 3.8), so for all $1 \leq r \leq (N-1)$ the prime diagonal of central rhombus r is conjugate to that of rhombus $(N-r)$ under reflection in σ . Having established the reflection symmetry for the prime diagonals, we note that the second diagonal of each central rhombus is a line segment centred at c and perpendicular to the prime diagonal of that rhombus, so the second diagonals of rhombi r and $(N-r)$ are also conjugate under reflection in σ . This is sufficient to show that central rhombi r and $(N-r)$ are conjugate under the reflection, since the vertices of these two rhombi are conjugate, being the end-points of the two diagonals.

The angle of inclination of the axis σ from the upward vertical follows simply from its definition as the bisector of $\angle 1^+c(N-1)^-$. Angle $\angle vc1^+ = \frac{1}{2}\theta$ and $\angle vc(N-1)^- = \frac{1}{2}\theta + (N-2)\theta + \pi$, so

$$\angle vc\sigma = \frac{1}{2}(\angle vc1^+ + \angle vc(N-1)^-) = \frac{1}{2}(N-1)\theta + \frac{\pi}{2}. \quad (3.20)$$

Finally, to show that the eccentric rhombi are conjugate under reflection in σ we use that A' and Z' are translations of the extra central rhombi A and Z , and that the latter are conjugate under the reflection. If we let M denote reflection in σ and H_A and H_Z denote the translations taking A to A' and Z to Z' respectively, it is sufficient to show that $MH_A = H_ZM$, for then $MA' = MH_AA = H_ZMA = H_ZZ = Z'$. To show $MH_A = H_ZM$ we introduce coordinates with the origin at c ; M is then linear so for any point x ,

$$MH_Ax = M(x - T^{-1}b) = Mx + M(-T^{-1}b), \quad (3.21)$$

and

$$H_ZMx = Mx + R_{(N-1)\theta}T^{-1}b, \quad (3.22)$$

and the points $-T^{-1}b$ and $R_{(N-1)\theta}T^{-1}b$ are conjugate under reflection in σ because angle $\angle_{vc}(-T^{-1}b) = \pi$ and $\angle_{vc}(R_{(N-1)\theta}T^{-1}b) = (N-1)\theta$, so their angle bisector makes an angle $\frac{1}{2}(N-1)\theta + \frac{\pi}{2} = \angle_{vc}\sigma$ with the upward vertical. \square

An illustration of the reflection symmetry of the rhombus configuration is shown in Figure 3.9; here the configuration, for $N = 4$ and $\theta = 0.5$, is one of those already encountered (as Figure 3.7(b) in the example above) but in this plot the axis of reflection σ has been added.

This reflection in the axis σ is the only symmetry present in the configuration of *all* the rhombi $A', 1, 2, \dots, (N-1), Z'$. If, however, we restrict attention to the central rhombi $1, 2, \dots, (N-1)$ alone, then there is a second symmetry, namely rotation by π about the point c . The reason is straightforward: any individual rhombus has a rotational symmetry by π about its centre, and all the central rhombi share the same centre point c . The combination of this rotation together with the reflection in σ gives a second reflection symmetry for the collection of central rhombi, in an axis through c perpendicular to σ .

Our strategy for describing the dependence of the common intersection \bar{S}' of rhombi $A', 1, 2, \dots, (N-1), Z'$ on the parameter θ begins by viewing the set \bar{S}' as arising from the

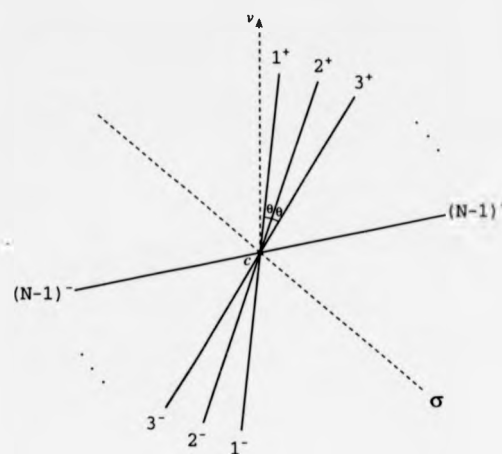


Figure 3.8: The arrangement of the prime diagonals of the central rhombi with respect to the axis of symmetry σ .

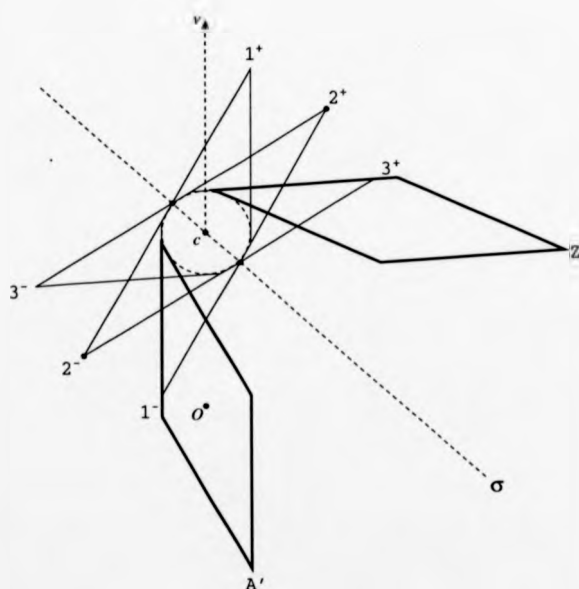


Figure 3.9: The location of the axis of reflection symmetry in a rhombus configuration for the string $+00+$.

intersection of two component parts. The first of these is a *central polygon* formed by the intersection of the central rhombi $1, 2, \dots, (N-1)$, and the second is the (possibly empty) intersection of the eccentric rhombi A' and Z' . From the paragraph above it follows that this central polygon possesses, in addition to the reflection in σ , the rotation and perpendicular reflection symmetries.

For the rest of this section we will focus on the position of the axis σ relative to the central polygon. An immediate consequence of the reflection symmetry for the central polygon is that a point where the symmetry axis σ meets the central polygon (there are two such points) must be either a vertex of the polygon, or the mid-point of a side perpendicular to the axis σ . Both possibilities can occur, and Figure 3.10 shows examples of the central polygon and the symmetry line for rhombus configurations corresponding to these two situations. (Note that the two points where σ meets the central polygon are either both vertices, or both mid-points of sides : this is an immediate consequence of the second reflection symmetry for the central polygon, mentioned above.) The next result allows us to say precisely which of the two situations arises for a given rhombus configuration specified by N and θ , and is an important tool in our subsequent analysis of when S' is non-empty.

Proposition 3.4

- (a) When N is odd the axis of reflection σ meets the central polygon at the mid-point of a side.
- (b) When N is even the axis σ meets the central polygon at a vertex, *except* for a finite set of isolated values of θ where σ meets the central polygon at the mid-point of a side.

Proof

For part (a), when N is odd, we consider the central rhombus $\frac{1}{2}(N-1)$. Referring back to Section 3.4, we see that one of its points of contact with the central circle makes an angle of $\frac{\pi}{2} + \frac{1}{2}(N-1)\theta$ from the vertical. This angle coincides with the angle of inclination of the axis σ , so σ meets central rhombus $\frac{1}{2}(N-1)$ precisely at this point of contact, and the side of

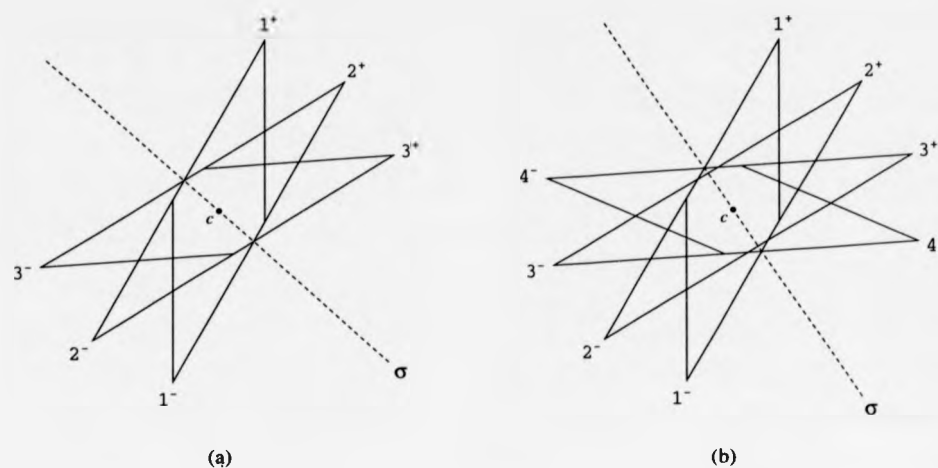


Figure 3.10: The axis of symmetry σ meets the central polygon at either (a) two vertices, or (b) the mid-points of two sides.

the rhombus is perpendicular to σ . The point of contact lies on the central polygon (since it is on the central circle contained within the central polygon), and cannot be a vertex of the central polygon (a vertex of the central polygon is formed as the intersection of two sides of central rhombi whose points of contact with the central circle are distinct), so must be the mid-point of a side.

When N is even we show that the intersection of σ with the central polygon is at a vertex by arguing that no side of any central rhombus is perpendicular to the axis σ . The angles of inclination of the sides of the $(N-1)$ central rhombi are $0, \theta, 2\theta, \dots, (N-1)\theta$. A side is perpendicular to σ when their angles (modulo π) differ by $\frac{\pi}{2}$, i.e. a side with angle $k\theta$ is perpendicular to σ at values of θ satisfying

$$k\theta + l\pi = \frac{1}{2}(N-1)\theta \quad (3.23)$$

for some integer l . Clearly for each $0 \leq k \leq (N-1)$, there are a finite number of isolated values of θ between 0 and π which satisfy (3.23); for all θ avoiding one of these exceptional values the intersection of σ with the central polygon occurs at a vertex. At an exceptional

value for θ , one of the central rhombi has two sides perpendicular to σ . It follows that the points of contact of these two sides are the two points of intersection of σ with the central circle, and then the argument above for the odd case shows that σ intersects the mid-point of a side of the central polygon. \square

Remark : In the case when N is even, the exceptional values of θ are isolated, so although σ passes through the mid-point of a side at an exceptional value of θ , immediately above or below this value the configuration must be such that σ passes through a vertex. This abrupt change in the configuration arises when two (or more) sides of the central rhombi (which for θ immediately above or below the exceptional value intersect to produce a vertex of the central polygon on σ) instantaneously become co-linear at the exceptional value so that the vertex disappears. The three plots in Figure 3.11 show an example of this behaviour; we use the string $+00+$ of length $N = 4$ again and plot the configuration below and above, and exactly at, the exceptional value $\theta = \frac{2\pi}{3}$.

We can list in full the exceptional values of θ at which the axis σ meets the central polygon at the mid-point of a side. Rearranging equation (3.23), we have

$$\theta = \frac{2l\pi}{(N-1)-2k} \quad (3.24)$$

for $k = 0, 1, \dots, (N-1)$ and all integers l such that $0 < \theta < \pi$. These values of θ may be listed as

$$2\pi \frac{p}{q} \quad \text{for} \quad q = 3, 5, 7, \dots, (N-1) \quad \text{and} \quad 1 \leq p \leq \frac{1}{2}(q-1), \quad (3.25)$$

and the exceptional values for short length strings $+0\dots 0+$ are given in the table below.

| String Length (N) | Exceptional Values of θ |
|-----------------------|--|
| 2 | no value |
| 4 | $\frac{2\pi}{3}$ |
| 6 | $\frac{2\pi}{3}, \frac{2\pi}{5}, \frac{4\pi}{5}$ |
| 8 | $\frac{2\pi}{3}, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}$ |
| 10 | $\frac{2\pi}{3}, \frac{2\pi}{5}, \frac{4\pi}{5}, \frac{2\pi}{7}, \frac{4\pi}{7}, \frac{6\pi}{7}, \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$ |

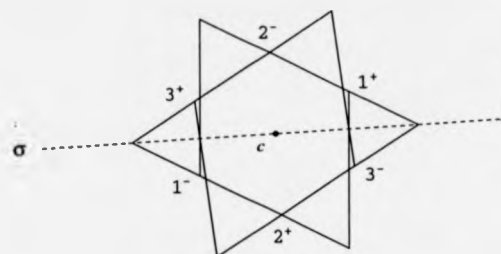
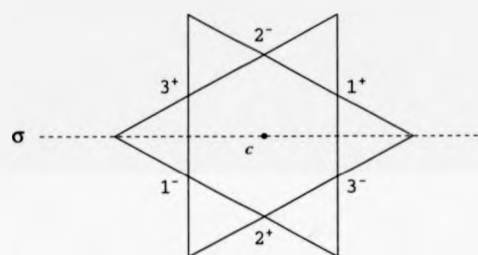
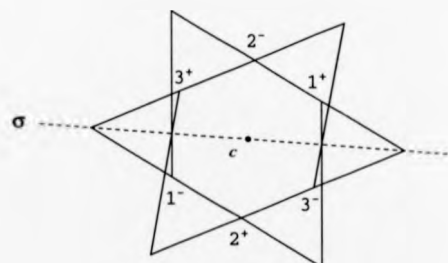
(a) θ just below $\frac{2\pi}{3}$ (b) $\theta = \frac{2\pi}{3}$ (c) θ just above $\frac{2\pi}{3}$

Figure 3.11: Plots of the rhombus configuration for the string $+00+$ around the exceptional value $\theta = \frac{2\pi}{3}$. Neighbouring sides of the central polygon become co-linear at the exceptional value, and then σ can pass through the mid-point of a side.

3.7 Determining when the rhombi have a common intersection

In this section we apply the ideas developed above which describe the locations of the rhombi and the symmetries of the configuration, to establish precisely when a common intersection of rhombi $A', 1, 2, \dots, (N-1), Z'$ occurs for a configuration specified in terms of the string length N and parameter θ . The key results in this section are Propositions 3.6 and 3.8, describing respectively the occurrences of a common intersection for the cases N odd and N even.

We shall adopt the viewpoint introduced in Section 3.6 that a common intersection of all the rhombi $A', 1, 2, \dots, (N-1), Z'$ arises as the overlap of the intersection of the two eccentric rhombi with the central polygon. Our first result, Proposition 3.5 below, lists the features associated with the intersection of A' and Z' , showing in particular that these two rhombi intersect when the parameter θ falls within any one of a number of disjoint intervals, all of which are easily specified in terms of N . Thereafter we may disregard values of θ outside of these intervals, because of course A' and Z' at least must intersect for there to be a common intersection of all the rhombi.

Proposition 3.5

The following four statements concerning the eccentric rhombi A' and Z' are equivalent :

- (i) Eccentric rhombi A' and Z' intersect.
- (ii) The axis of reflection symmetry intersects A' (or equivalently Z').
- (iii) θ belongs to one of the intervals $\left[\frac{(2k-1)\pi}{N}, \frac{2k\pi}{N}\right]$, where $k = 1, 2, \dots, \left\lfloor \frac{N}{2} \right\rfloor$.
- (iv) The angle (modulo 2π) of the point of contact of Z' is between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$.

Proof

(i) \Leftrightarrow (ii) The result follows simply from the fact that A' and Z' are conjugate under reflection in σ . If σ does not intersect A' (equivalently Z'), then A' and Z' have no common intersection because they lie in opposite halves of the plane, separated by the axis σ .

Conversely, if σ intersects A' (or Z') then the part of the axis σ which intersects A' (respectively Z') also belongs to Z' (respectively A').

(ii) \Leftrightarrow (iii) Because of the location of the point c with respect to rhombus A' , the symmetry axis σ intersects A' if and only if it intersects the upper edge of the rhombus A' . We can specify the range of angles of inclination for the symmetry axis to intersect this upper edge: the extremes of the ranges are when the axis passes through the two end-points of the upper edge, i.e. the upper two vertices of rhombus A' . The angles of inclination of σ when it meets these vertices are $\pi - \frac{1}{2}\theta$ and $\frac{3\pi}{2} - \frac{1}{2}\theta$; this follows because the vertices in question are also vertices of central rhombus A and we saw in Section 3.4 that the angle of inclination of the prime diagonal of rhombus A is $-\frac{1}{2}\theta$. Thus the symmetry line intersects A' at values of θ between 0 and π satisfying

$$\pi - \frac{1}{2}\theta \leq \frac{1}{2}(N-1)\theta + \frac{\pi}{2} + l\pi \leq \frac{3\pi}{2} - \frac{1}{2}\theta \quad (3.26)$$

for some integer l , which may be rearranged, setting $k = 1 - l$, to give

$$\frac{(2k-1)\pi}{N} \leq \theta \leq \frac{2k\pi}{N}, \quad k = 1, 2, \dots, \left\lfloor \frac{N}{2} \right\rfloor. \quad (3.27)$$

(iii) \Leftrightarrow (iv) The position angle of the point of contact of rhombus Z' is $\frac{3\pi}{2} + N\theta$ which, for θ in the range $\frac{(2k-1)\pi}{N} \leq \theta \leq \frac{2k\pi}{N}$, varies from $\frac{3\pi}{2} + (2k-1)\pi$ to $\frac{3\pi}{2} + 2k\pi$, i.e. $\frac{\pi}{2}$ to $\frac{3\pi}{2}$ when reduced modulo 2π . \square

We see from this result that, for a string $+0\dots 0+$ of length N , rhombi A' and Z' intersect when θ is in any one of $\left\lfloor \frac{N}{2} \right\rfloor$ disjoint intervals. For the particular string used in the example of Section 3.5, namely $+00+$ of length $N = 4$, these intervals are $\left[\frac{\pi}{4}, \frac{\pi}{2}\right]$ and $\left[\frac{3\pi}{4}, \pi\right)$. Of the plots of the rhombus configurations in Figure 3.7, those corresponding to values of θ within one of these intervals are Figures 3.7(d) to 3.7(i) for the first intersection of A' and Z' , and Figures 3.7(m) to 3.7(p) for the second intersection. All values of θ outside of these intervals give rise to configurations where rhombus Z' is located around the top half of the central circle and consequently does not intersect A' .

It will be helpful to introduce new terminology to refer to the intervals for θ so that A' and Z' intersect, since the configurations corresponding to values of θ in one of these intervals now become the focus of our investigation. We will call the interval $\left[\frac{(2k-1)\pi}{N}, \frac{2k\pi}{N} \right]$ the k^{th} pass, and the integer $1 \leq k \leq \left\lfloor \frac{N}{2} \right\rfloor$ the *pass number*, these names deriving from the motion of rhombus Z' which, as θ increases, repeatedly swings around the central circle each time passing through rhombus A' . We can also make precise the idea of the rhombus Z' "swinging around" the central circle by viewing the evolution of rhombus Z' in terms of the movement of its point of contact around the central circle. The rhombus Z' makes $\frac{N}{2}$ full revolutions around the central circle as θ increases through its whole range from 0 to π , and from the start of one pass to the start of the next, the point of contact of Z' rotates exactly once around the central circle.

The task ahead is to describe which are the θ values within a given pass that give rise to a configuration having a common intersection of all the rhombi $A', 1, \dots, (N-1), Z'$. Just as we found that the kind of boundary point of the central polygon where the symmetry line meets it depends on whether the string length N is odd or even, so the two cases, N odd and N even, differ appreciably in the manner in which a common intersection of all the rhombi arises. We thus treat the two cases separately, starting with N odd, which is much the easier case : when N is odd there is a common intersection at only one isolated θ value within each pass.

Proposition 3.6 (Dependence of the common intersection on θ when N is odd)

When N is odd, the common intersection of rhombi $A', 1, 2, \dots, (N-1), Z'$ is either empty or is simply a line segment. The latter case occurs only at the isolated values $\theta = \frac{2k\pi}{N+1}$, $k = 1, 2, \dots, \frac{1}{2}(N-1)$.

Proof

Because A' and Z' are conjugate under reflection in σ , their intersection is either empty or a convex polygon, symmetric with respect to reflection in σ . By symmetry and convexity, if the intersection of A' and Z' meets the central polygon then this common intersection must include a point on the axis σ . From Proposition 3.4, σ intersects the central polygon at

the mid-points of two sides, so the portion of σ within the central polygon is a diameter of the central circle. On the other hand, the eccentric rhombi A' and Z' lie outside the central circle. Thus the only point on σ which can belong to all $(N+1)$ rhombi is the point where σ meets the lower half of the central circle.

A common intersection at the point where σ meets the lower half of the central circle will only occur if this point is simultaneously the point of contact of both A' and Z' . The angle of the point of contact of A' is $\frac{3\pi}{2} - \theta$, and of the symmetry line is $\frac{1}{2}(N-1)\theta + \frac{\pi}{2}$, so the point of contact is on the axis of symmetry if

$$\frac{3\pi}{2} - \theta = \frac{1}{2}(N-1)\theta + \frac{\pi}{2} + l\pi \quad (3.28)$$

for some integer l . After rearranging we find that the relevant values of θ between 0 and π are $\theta = \frac{2k\pi}{N+1}$, $k = 1, 2, \dots, \frac{1}{2}(N-1)$. For any one of these values the common intersection is a line-segment, namely that portion of the coincident edges of A' and Z' which is also shared with the side of the central polygon bisected by the axis σ . \square

The case when N is even is appreciably more complicated, and here interesting behaviour arises. Within each pass there may be either a single θ value or a whole interval for θ throughout which there is a common intersection. The key step in the argument in this case is the observation that the occurrence of a common intersection of all the rhombi corresponds to a particular ordering amongst the points of contact. This fact is established in Proposition 3.7 below.

Once again it will be important to make a distinction between the exceptional values of θ , listed in (3.25), and all other values. At a non-exceptional value the symmetry axis σ does not pass through a point of contact of one of the central rhombi; thus there is a nearest point of contact on the central circle to either side of the point of intersection of σ with the lower half of the central circle. We call these points of contact p_- and p_+ (the labels are applied so that the position angle of p_- is less than that of p_+). Figure 3.12(a) shows the

location of p_- and p_+ with respect to the symmetry axis and the central circle for a typical, non-exceptional value of θ . Note that p_- and p_+ are conjugate under the reflection in σ , so p_+ is as much in advance of σ as p_- is behind. At an exceptional value, the special feature of the rhombus configuration is that the symmetry line passes through a point of contact of a central rhombus; so now the two points p_- and p_+ coincide with the point where σ intersects the lower half of the central circle. Figure 3.12(b) is the appropriate diagram for one of these exceptional values.

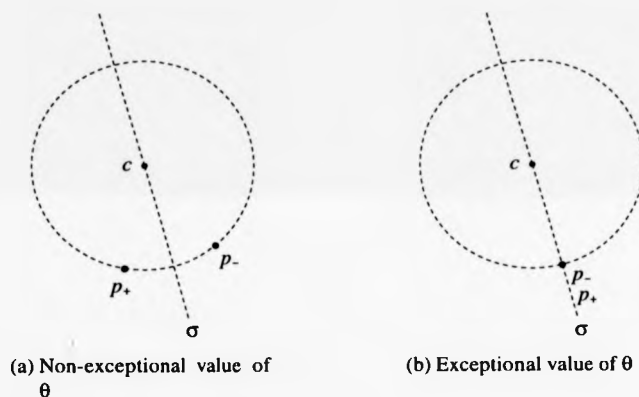


Figure 3.12: The locations of p_- and p_+ , the closest points of contact of the central rhombi to the axis σ .

Proposition 3.7

The common intersection of rhombi $A', 1, 2, \dots, (N-1), Z'$ is non-empty if and only if the points of contact $p_{A'}$ and $p_{Z'}$ of rhombi A' and Z' are coincident with, or lie between p_- and p_+ .

Proof

We first consider the case where $p_{A'}$ and $p_{Z'}$ lie outside the shorter arc joining p_- to p_+ . Let q be the point of intersection of the lines tangent to the central circle at p_- and p_+ . Because

of reflection symmetry, the point q lies on the lower half of the axis of symmetry σ . We may assume that A' and Z' have a non-empty intersection (otherwise there is nothing to prove) and this means that both $p_{A'}$ and $p_{Z'}$ lie on the lower half of the central circle. Consequently the lines tangent to the central circle to $p_{A'}$ and $p_{Z'}$ will intersect at a point q' on the lower half of the symmetry axis σ . When $p_{A'}$ and $p_{Z'}$ are outside of the arc joining p_- to p_+ , the point q' lies further from the centre c of the central circle than q . Figure 3.13(a) illustrates the situation; note that dependent on the value of θ , the labelling of $p_{A'}$ and $p_{Z'}$ may be interchanged, but this is of no consequence because it would not affect the location of q' .

Let l and l' be the lines perpendicular to σ through q and q' respectively. The common intersection of A' and Z' lies on the opposite side of the lines extending $q'p_{A'}$ and $q'p_{Z'}$ to the central circle, so lies on the opposite side of l' to the central circle. But since p_-qp_+ is a part of the perimeter of the central polygon, by convexity the central polygon lies on the same side of l as the central circle. Thus the intersection of A' and Z' does not meet the central polygon.

For the converse we must show that if $p_{A'}$ and $p_{Z'}$ lie between p_- and p_+ then there is a common intersection of all the rhombi. The first step is to show that there is an intersection between the two eccentric rhombi, A' and Z' . We saw in Proposition 3.5 that A' and Z' intersect if and only if the angle of $p_{Z'}$ (taken modulo 2π) lies between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$. But central rhombus 1 has two points of contact which remain fixed at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$, so a point of contact of rhombus 1 lies between $p_{A'}$ and $p_{Z'}$ whenever A' and Z' do not intersect.

Since A' and Z' are conjugate under the reflection in σ , whenever A' and Z' intersect, the point q' where the lines tangent to the central circle at $p_{A'}$ and $p_{Z'}$ meet the symmetry axis, belongs to this intersection. If $p_{A'}$ and $p_{Z'}$ lie between p_- and p_+ , the point q' is in the interior of the triangle $\triangle p_-p_+q$, and hence in the interior of the central polygon. Figure 3.13(b) is the diagram for the case when $p_{A'}$ and $p_{Z'}$ lie between p_- and p_+ , and shows the point q' , this time inside the central polygon. We may actually say more in this case: the intersection of rhombi $A', 1, 2, \dots, (N-1), Z'$ is a set with non-empty interior.

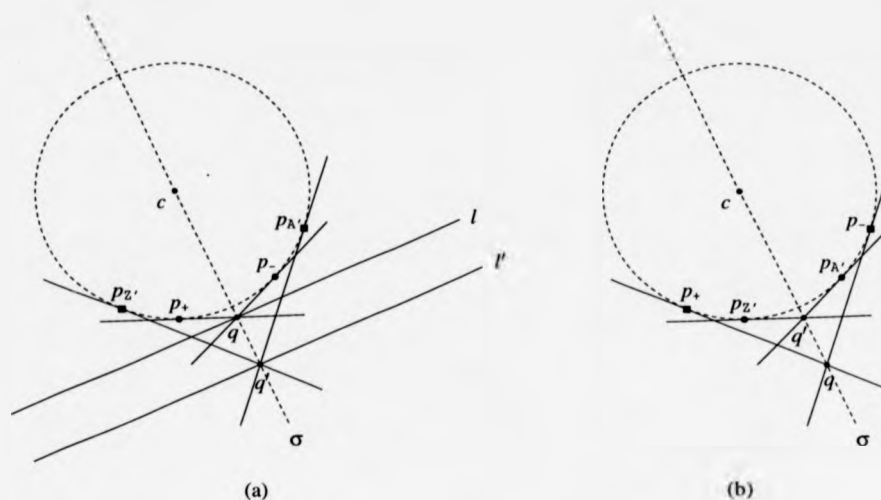


Figure 3.13: Illustrations for the proof of Proposition 3.7. The region common to the eccentric rhombi intersects the central polygon only when p_- and p_+ lie between p_A and p_Z .

In the case when p_A and p_Z coincide with p_- and p_+ , the points q and q' are coincident and it is only points on the line l (now coincident with l') which could belong to a common intersection. There are two possibilities: when p_- and p_+ are distinct, q is the only point of the central polygon on the line l , so the common intersection of rhombi $A', 1, 2, \dots, (N-1), Z'$ consists of the point q alone. Otherwise, when p_- and p_+ are coincident (at an exceptional value for θ), one side of the central polygon lies along part of the line l , as do a side of each of A' and Z' ; thus the common intersection is a segment of the line l (which is now tangent to the central circle). \square

The importance of this result is that, by relating the occurrences of a common intersection of rhombi $A', 1, 2, \dots, (N-1), Z'$ to a condition purely in terms of the points of contact, we have simplified the problem of following the evolution of a common intersec-

tion by reducing it to the much easier task of understanding the relative movement of the points of contact. This latter task is more straightforward because each point of contact simply rotates clockwise at a fixed rate around the central circle as θ is increased. We are now in a position to present our result specifying when there is a common intersection of all the rhombi in the case of even length strings.

Proposition 3.8 (Dependence of the common intersection on θ when N is even)

When N is even, within each interval $\left[\frac{(2k-1)\pi}{N}, \frac{2k\pi}{N}\right]$ of θ where the eccentric rhombi intersect (i.e. within each pass), either

- (a) there is a single closed interval for θ in which rhombi $A', 1, 2, \dots, (N-1), Z'$ have non-empty intersection, and this interval includes the value $\theta = \frac{2k\pi}{N+1}$ strictly within it,

or

- (b) there is a single isolated value $\theta = \frac{2k\pi}{N+1}$ at which the common intersection of the rhombi consists of a line segment. Either side of this value the common intersection is empty.

The degenerate case (b) occurs precisely when the pass number k , and $(N+1)$ have a common factor.

Proof

We restrict attention to the k^{th} pass, and let $\theta = \frac{2k\pi}{N+1}$. This value is in the interior of the interval for the k^{th} pass since $\frac{(2k-1)\pi}{N} < \frac{2k\pi}{N+1} < \frac{2k\pi}{N}$. At $\theta = \frac{2k\pi}{N+1}$ the points of contact of A' and Z' coincide because

$$\text{position angle of } p_{A'} = \frac{3\pi}{2} - \theta = \frac{3\pi}{2} - \frac{2k\pi}{N+1},$$

$$\text{position angle of } p_{Z'} = \frac{3\pi}{2} + N\theta = \frac{3\pi}{2} + \frac{2Nk\pi}{N+1},$$

from which we can see that the difference between these angles is a multiple of 2π . Furthermore, because $p_{A'}$ and $p_{Z'}$ are conjugate under the reflection in σ , they coincide at the point where σ meets the lower half of the central circle. For this particular value of θ , all rhombi $A', 1, 2, \dots, (N-1), Z'$ have a common intersection because the coincident point, being a

point of the central circle, also belongs to each of the central rhombi. To understand what happens to this common intersection as θ moves away from $\frac{2k\pi}{N+1}$, we need to distinguish two cases, which correspond to the two possibilities isolated in (b) of Proposition 3.4.

(a) $\theta = \frac{2k\pi}{N+1}$ is not an exceptional value

When $\theta = \frac{2k\pi}{N+1}$ is not an exceptional value, the coincident points of contact $p_{A'}$ and $p_{Z'}$ lie on the arc of the central circle between p_- and p_+ . If we increase θ by an amount α , the angle of $p_{Z'}$ increases by $N\alpha$, whereas the angle of p_+ increases by at most $(N-1)\alpha$, because p_+ is one of the points of contact of the central rhombi. Consequently as θ increases beyond the value $\frac{2k\pi}{N+1}$, the arc of the central circle between $p_{Z'}$ and p_+ contracts, until a critical value θ_E is reached where $p_{Z'}$ and p_+ coincide. Likewise the length of the arc between $p_{A'}$ and p_- decreases until they become coincident at θ_E because $p_{A'}$ and p_- are the respective reflections of $p_{Z'}$ and p_+ in the axis σ . For all values of θ above θ_E until the end of the k^{th} pass at $\theta = \frac{2k\pi}{N}$, p_- and p_+ lie on the arc of the circle between $p_{A'}$ and $p_{Z'}$, the separation between $p_{Z'}$ and p_+ (and between $p_{A'}$ and p_-) increasing with θ . Figure 3.14 illustrates this evolution of the arrangement of the points of contact $p_{A'}$, $p_{Z'}$, p_- and p_+ as θ increases. Initially, when $\theta = \frac{2k\pi}{N+1}$, $p_{A'}$ and $p_{Z'}$ lie coincident on the symmetry axis σ between p_- and p_+ . As θ increases $p_{A'}$ and $p_{Z'}$ approach p_- and p_+ , and at $\theta = \theta_E$, $p_{A'}$ meets p_- and $p_{Z'}$ meets p_+ . Increasing θ further, $p_{A'}$ and $p_{Z'}$ "overtake" p_- and p_+ .

The evolution of the arrangement of the points of contact is entirely similar when we decrease θ from $\frac{2k\pi}{N+1}$; reducing θ by an amount α , the angle of $p_{Z'}$ decreases by $N\alpha$ whilst the angle of p_- decreases by at most $(N-1)\alpha$. So now there is a critical value θ_B where $p_{Z'}$ coincides with p_- and $p_{A'}$ coincides with p_+ ; for $\theta_B < \theta < \frac{2k\pi}{N+1}$, $p_{A'}$ and $p_{Z'}$ lie between p_- and p_+ , and from the beginning of the k^{th} pass to θ_B , p_- and p_+ lie between $p_{A'}$ and $p_{Z'}$.

We now apply Proposition 3.7 to relate the arrangement of the points of contact to the common intersection of rhombi $A', 1, 2, \dots, (N-1), Z'$. The outcome is that for $\theta_B < \theta < \theta_E$, the rhombi have a common intersection with non-empty interior, at $\theta = \theta_B$ and $\theta = \theta_E$ the common intersection is a single point, and for all other values of θ within the current pass there is no common intersection of rhombi $A', 1, 2, \dots, (N-1), Z'$.

(b) $\theta = \frac{2k\pi}{N+1}$ is an exceptional value

When $\theta = \frac{2k\pi}{N+1}$ is an exceptional value, the four points $p_{A'}$, $p_{Z'}$, p_- and p_+ are all coincident, and as was shown in the proof of Proposition 3.7, the common intersection of all the rhombi lies along a segment of the line tangent to the unit circle at this coincident point. Increasing θ by an amount α , the position angle of $p_{Z'}$ increases by $N\alpha$ while that of p_+ increases by an amount $\leq (N-1)\alpha$; consequently for all $\theta > \frac{2k\pi}{N+1}$ within the k^{th} pass, the arrangement of points of contact is that $p_{A'}$ and $p_{Z'}$ enclose p_- and p_+ . Likewise $p_{A'}$ and $p_{Z'}$ enclose p_- and p_+ for all $\theta < \frac{2k\pi}{N+1}$ within the k^{th} pass. By Proposition 3.7, it follows that there is no common intersection of the rhombi for θ within the k^{th} pass, except at the isolated value $\theta = \frac{2k\pi}{N+1}$.

The degenerate case (b) arises whenever $\frac{2k\pi}{N+1}$ is one of the exceptional values (3.25), namely

$$2\pi \frac{p}{q} \quad \text{for} \quad q = 3, 5, 7, \dots, (N-1) \quad \text{and} \quad 1 \leq p \leq \frac{1}{2}(q-1).$$

This occurs precisely when the pass number k and $N+1$ have a common factor, for if k and $(N+1)$ share a common factor d (necessarily ≥ 3), after dividing through the numerator and denominator by d we obtain a fraction $\frac{p}{q}$ with denominator $q = \frac{N+1}{d} \leq (N-1)$. We must also demonstrate that $p \leq \frac{1}{2}(q-1)$, and this follows because $k \leq \frac{N}{2} = \frac{1}{2}((N+1)-1)$; dividing through by d gives $p \leq \frac{1}{2}(q - \frac{1}{d})$, but p is an integer so $p \leq \frac{1}{2}(q-1)$. Conversely if k and $(N+1)$ are coprime then the irreducible denominator $(N+1)$ is larger than the greatest denominator for an exceptional value, namely $(N-1)$, so $\frac{2k\pi}{N+1}$ does not equal any of the exceptional values. \square

Remark : It should be noted that p_+ in the proof need not remain the *same* point of contact while θ increases from $\frac{2k\pi}{N+1}$ to its value $\frac{2k\pi}{N}$ at the end of a pass. It is entirely possible that at some value of θ , p_+ becomes coincident with a point of contact of another central rhombus, thereafter p_+ will be identified with this point of contact. (In terms of rates of rotation, the original p_+ may catch up with a slower moving point of contact and then become this

slower moving point; thus the rate of rotation of p_+ around the central circle need not be constant, diminishing by a discrete jump whenever a slower moving point of contact is overtaken.) This circumstance does not affect the argument since we used only that the increase in the position angle of p_+ is $\leq (N-1)\alpha$ when θ is increased by α , which is true regardless of whether p_+ switches between points of contact as θ increases to $\theta + \alpha$.

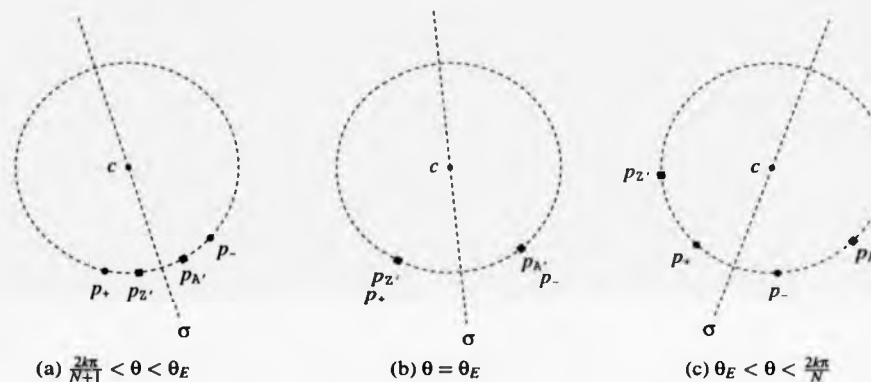


Figure 3.14: The changes in the arrangement of the points of contact $p_{A'}$ and $p_{Z'}$ with respect to p_- and p_+ when θ is increased from $\frac{2k\pi}{N+1}$ to $\frac{2k\pi}{N}$.

The two central results from this section, Propositions 3.6 and 3.8, have allowed us to specify the form of the set of values of θ at which there is a common intersection of rhombi $A', 1, 2, \dots, (N-1), Z'$. To complete the proof of Proposition 3.1, all that remains is to calculate the end-points of the intervals $[\theta_B, \theta_E]$ arising for even length strings, and this calculation we describe in the next section.

We conclude with two examples which exhibit an interesting and surprising feature of the set of values of θ corresponding to a common intersection of all the rhombi. Namely that for the even length strings $+0 \dots 0+$, where typically the common intersections occur for sub-intervals of θ , certain of these sub-intervals are “missing”, and instead the common intersection occurs for just an isolated value of θ .

Example 3.4 : The string $+0 \dots 0+$ of length $N = 44$

Our first example is a string for which several of the passes include isolated values of θ rather than the sub-intervals over which we expect a common intersection of all the rhombi. Proposition 3.8 states that a common intersection at a single point occurs for the k^{th} pass precisely when k and $(N + 1)$ share a common factor. Here $N + 1 = 45 = 3^2 \cdot 5$, which has several factors; consequently a relatively high proportion of the $\frac{N}{2} = 22$ passes correspond to a common intersection at just a single θ value. Table 3.3 lists the pass number k and the nature of the set of θ values within that pass for which the rhombus configuration has non-empty intersection.

Example 3.5 : Counting the passes including a sub-interval of common intersection

The fact that there is a simple divisibility criterion that governs whether a common intersection of the rhombi occurs for an interval of θ values or just at a single value, means that a concise formula, in terms of the Euler totient function ϕ , can be given for the number of passes in which a common intersection occurs for an interval of θ values.

The number of passes which contain an interval of common intersection is the count of integers from 1 to $\frac{N}{2}$, inclusive, that are coprime to $(N + 1)$. Note that the count of integers coprime to $(N + 1)$ in the range $1, 2, \dots, \frac{N}{2}$ is the same as the count of numbers coprime to $(N + 1)$ in the range $\frac{N}{2} + 1, \frac{N}{2} + 2, \dots, N$ because the integer k is coprime to $(N + 1)$ if and only if $(N + 1 - k)$ is. Consequently the number of passes which contain an interval of common intersection is $\frac{1}{2}\phi(N + 1)$, with the remaining $\frac{1}{2}(N - \phi(N + 1))$ passes having a common intersection at a single isolated θ value. These values may be easily calculated from the prime factorisation of $(N + 1)$ via the formula

$$\phi(N + 1) = (N + 1) \prod_{\substack{p|(N+1) \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right). \quad (3.29)$$

Table 3.4 lists the number of passes where there is a common intersection for a sub-interval, and at an isolated value, for all even length strings $+0 \dots 0+$ up to length $N = 50$.

Note that for any string for which $(N + 1)$ is prime, the degenerate case of Proposition 3.8 never arises, and there is a sub-interval of common intersection within every pass.

| Pass Number (k) | $\text{hcf}(k, N+1)$ | Values of θ Corresponding to a Common Intersection |
|------------------------|----------------------|---|
| 1 | 1 | sub-interval of $[\frac{\pi}{44}, \frac{\pi}{22}]$ |
| 2 | 1 | sub-interval of $[\frac{3\pi}{44}, \frac{\pi}{11}]$ |
| 3 | 3 | single point $\frac{2\pi}{15}$ |
| 4 | 1 | sub-interval of $[\frac{7\pi}{44}, \frac{2\pi}{11}]$ |
| 5 | 5 | single point $\frac{2\pi}{9}$ |
| 6 | 3 | single point $\frac{4\pi}{15}$ |
| 7 | 1 | sub-interval of $[\frac{13\pi}{44}, \frac{7\pi}{22}]$ |
| 8 | 1 | sub-interval of $[\frac{15\pi}{44}, \frac{4\pi}{11}]$ |
| 9 | 9 | single point $\frac{2\pi}{5}$ |
| 10 | 5 | single point $\frac{4\pi}{9}$ |
| 11 | 1 | sub-interval of $[\frac{21\pi}{44}, \frac{\pi}{2}]$ |
| 12 | 3 | single point $\frac{8\pi}{15}$ |
| 13 | 1 | sub-interval of $[\frac{25\pi}{44}, \frac{13\pi}{22}]$ |
| 14 | 1 | sub-interval of $[\frac{27\pi}{44}, \frac{7\pi}{11}]$ |
| 15 | 15 | single point $\frac{2\pi}{3}$ |
| 16 | 1 | sub-interval of $[\frac{31\pi}{44}, \frac{8\pi}{11}]$ |
| 17 | 1 | sub-interval of $[\frac{3\pi}{4}, \frac{17\pi}{22}]$ |
| 18 | 9 | single point $\frac{4\pi}{5}$ |
| 19 | 1 | sub-interval of $[\frac{37\pi}{44}, \frac{19\pi}{22}]$ |
| 20 | 1 | sub-interval of $[\frac{39\pi}{44}, \frac{10\pi}{11}]$ |
| 21 | 3 | single point $\frac{14\pi}{15}$ |
| 22 | 1 | sub-interval of $[\frac{43\pi}{44}, \pi]$ |

Table 3.3: The 22 passes for the string $+0\dots 0+$ of length $N = 44$, listing for each whether a non-empty intersection of the rhombi occurs at an isolated value or over a sub-interval.

| String Length (N) | Number of Passes with an Isolated Value of Common Intersection $\frac{1}{2}(N - \phi(N + 1))$ | Number of Passes with a Sub-interval of Common Intersection $\frac{1}{2}\phi(N + 1)$ |
|--------------------------|---|--|
| 2 | 0 | 1 |
| 4 | 0 | 2 |
| 6 | 0 | 3 |
| 8 | 1 | 3 |
| 10 | 0 | 5 |
| 12 | 0 | 6 |
| 14 | 3 | 4 |
| 16 | 0 | 8 |
| 18 | 0 | 9 |
| 20 | 4 | 6 |
| 22 | 0 | 11 |
| 24 | 2 | 10 |
| 26 | 4 | 9 |
| 28 | 0 | 14 |
| 30 | 0 | 15 |
| 32 | 6 | 10 |
| 34 | 5 | 12 |
| 36 | 0 | 18 |
| 38 | 7 | 12 |
| 40 | 0 | 20 |
| 42 | 0 | 21 |
| 44 | 10 | 12 |
| 46 | 0 | 23 |
| 48 | 3 | 21 |
| 50 | 9 | 16 |

Table 3.4: The number of passes corresponding to a sub-interval or an isolated value of non-empty intersection of the rhombi, tabulated for strings $+0\dots 0+$ of even length $N \leq 50$.

3.8 Calculating the sub-intervals of common intersection

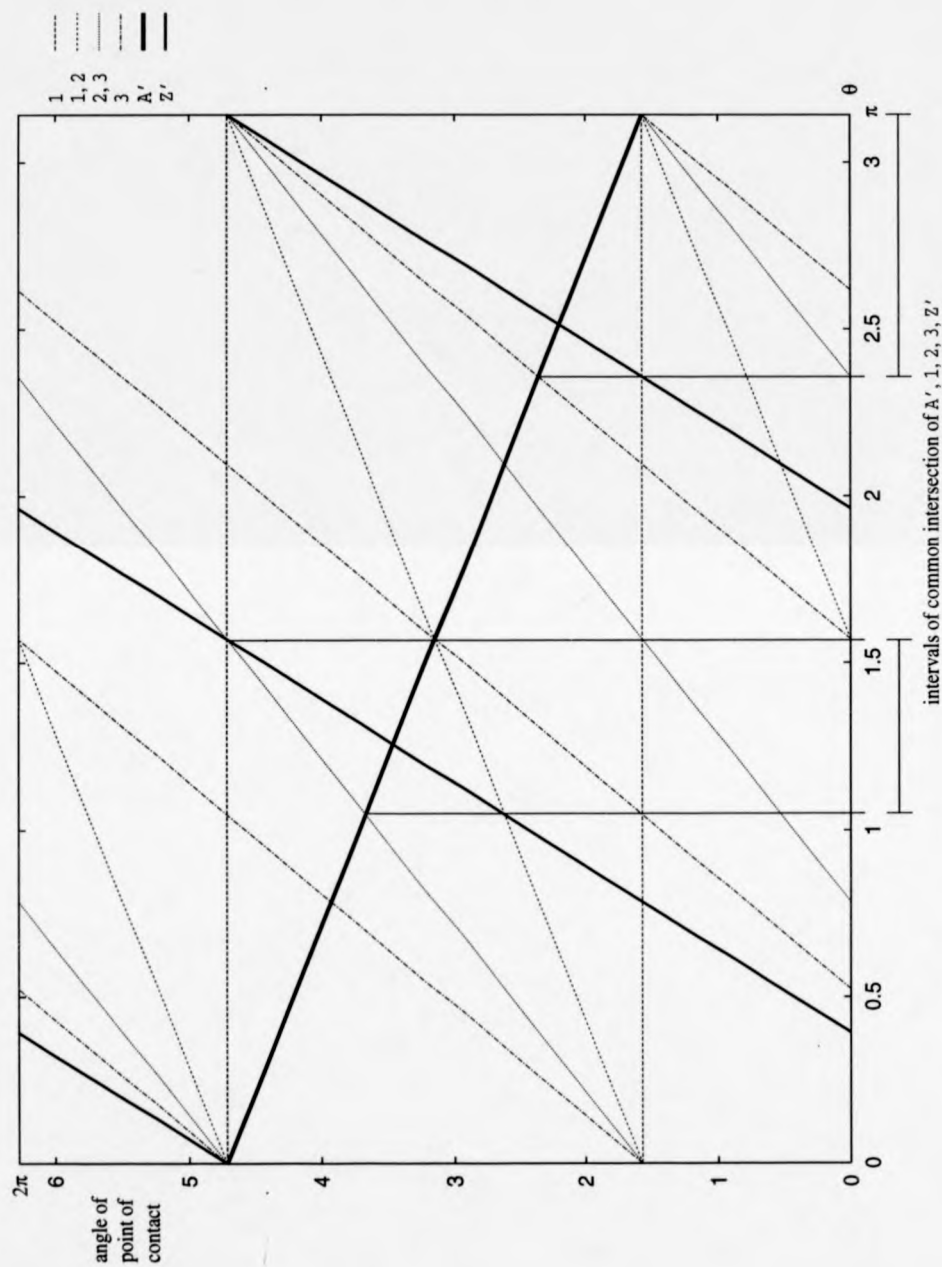
In this section we consider even length strings, for which we have shown there is a sub-interval $[\theta_B, \theta_E]$ of the k^{th} pass, whenever k and $(N+1)$ are coprime, where rhombi $A', 1, 2, \dots, (N-1), Z'$ have a common intersection. Before describing how θ_B and θ_E can be determined in general, we will give an example of a direct attempt to find their values. Although the approach is only suitable for short strings we include it here, as an example, because it illustrates ideas encountered in the last section.

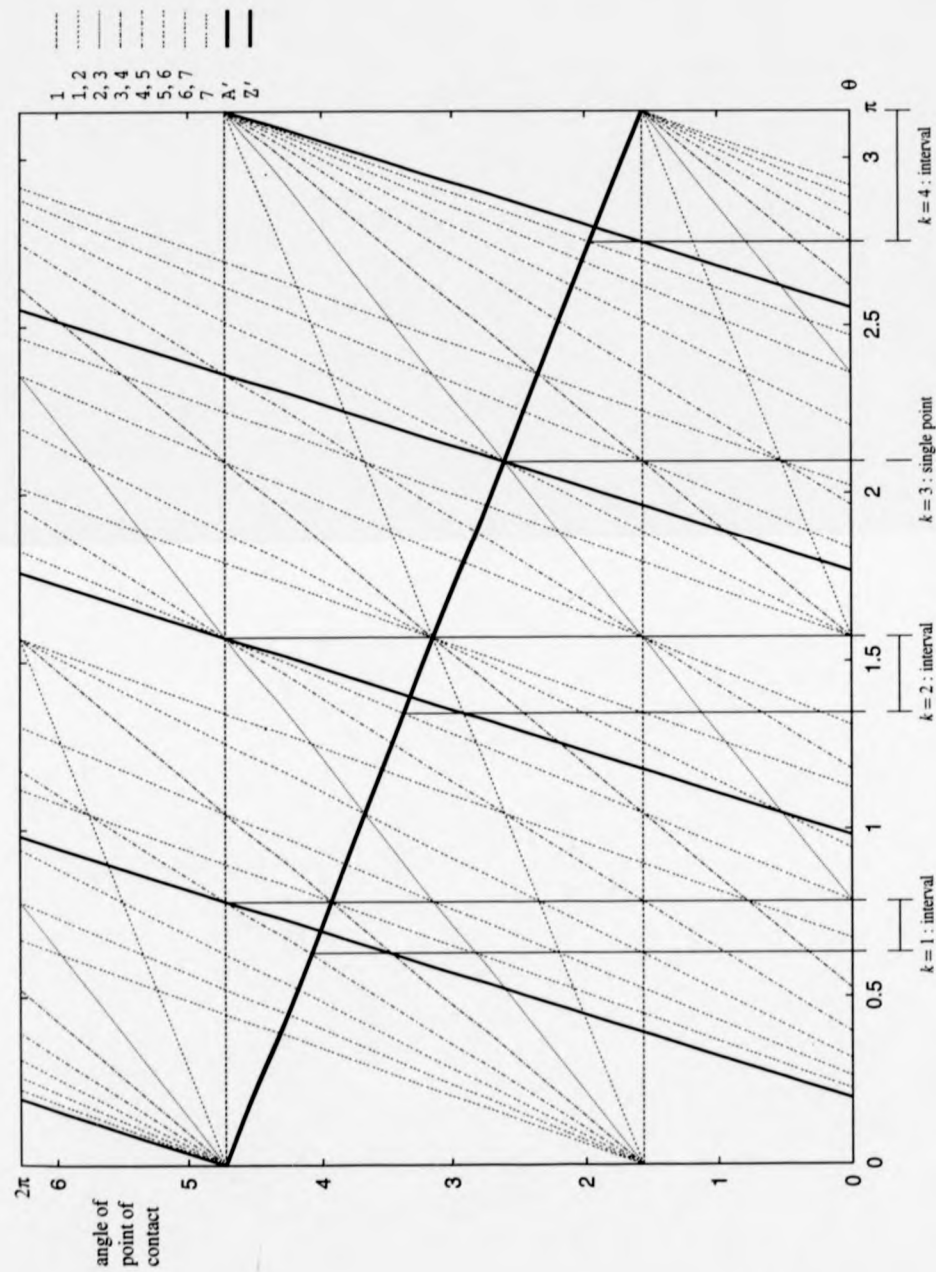
Example : Locating θ_B and θ_E by graphing the position angles of the points of contact

For short length strings, we may use Proposition 3.7 (i.e. that $p_{A'}$ and $p_{Z'}$ lie between p_- and p_+ precisely when there is a common intersection) as the basis of a graphical method for locating the sub-intervals $[\theta_B, \theta_E]$. We plot the position angles of each of the points of contact of the central rhombi, and of $p_{A'}$ and $p_{Z'}$ over the range 0 to π . The graph for each point has constant slope, but jumps down to zero whenever the position angle reaches 2π . Figure 3.15 is the plot corresponding to the string $+00+$; there each line has been labelled with the rhombus to which the point of contact belongs.

The special values $\theta = \frac{2k\pi}{N+1}$, where the points $p_{A'}$ and $p_{Z'}$ become coincident, and which played a central role in the proof of Proposition 3.8, are where the lines for $p_{A'}$ and $p_{Z'}$ meet. We know from the theory that there is a common intersection at $\theta = \frac{2k\pi}{N+1}$ with $p_{A'}$ and $p_{Z'}$ lying between p_- and p_+ . To locate θ_B , we decrease θ from this value, moving leftwards in the plot, until a graph for a point of contact of a central rhombus meets the graph for $p_{A'}$ (and of course, at this same value a graph for one of the other points of contact meets the graph for $p_{Z'}$). Once we have identified which point of contact is involved, the precise value of θ_B can be found by solving for the intersection of the two graphs. The upper end-point θ_E may be found in like manner by looking to the right of $\frac{2k\pi}{N+1}$.

To demonstrate the method we will find the sub-interval of common intersection within the first pass for the string $+00+$. With reference to Figure 3.15, moving left from the first intersection of the graphs labelled A' and Z' , we see that the line for A' first meets a line

Figure 3.15: A graph of the position angles of the points of contact for the string $+00+$.

Figure 3.16: The corresponding graph for the length $N = 8$ string $+0000000+$.

for a point of contact shared by central rhombi 2 and 3. From the list of the position angles in Section 3.4, the shared points of contact have angles $\frac{\pi}{2} + 2\theta$ and $\frac{3\pi}{2} + 2\theta$; by following the line back to $\theta = 0$ we can see that the graph in question is $\frac{\pi}{2} + 2\theta$. The point $p_{A'}$ has position angle $\frac{3\pi}{2} - \theta$, so θ_B is the solution to $\frac{\pi}{2} + 2\theta_B = \frac{3\pi}{2} - \theta_B$, i.e. $\theta_B = \frac{\pi}{3}$. To the left of $\frac{\pi}{3}$, lines corresponding to points of contact of central rhombi come between the lines A' and Z' ; this is as expected, because for these values of θ we know $p_{A'}$ and $p_{Z'}$ enclose the points p_- and p_+ . For the upper end-point, θ_E , we look to the right of where the lines A' and Z' intersect and note that the first intersection for A' is simultaneously with the line $\frac{\pi}{2} + \theta$ (for a point of contact shared by rhombi 1 and 2) and the line $\frac{3\pi}{2} + 3\theta - 2\pi$ (for a point of contact of rhombus 3). Solving for either intersection gives $\theta_E = \frac{\pi}{2}$.

Figure 3.15 contains an instance of the phenomenon remarked upon after the proof of Proposition 3.8, namely that p_- and p_+ need not remain identified with one point of contact, but may switch between several as θ is increased. Such a switch occurs at the value $\theta_E = \frac{\pi}{2}$ just identified: immediately to the left of $\frac{\pi}{2}$, p_- is a point of contact shared by rhombi 1 and 2, but at $\theta = \frac{\pi}{2}$ a point of contact of rhombus 3 (in the figure this is the steeper of the two lines) catches up with it, and immediately to the right of $\frac{\pi}{2}$, p_- identifies with this point of contact of rhombus 3. Figure 3.16 is the plot for the length $N = 8$ string $+000000+$ and is significantly more complicated on account of the additional points of contact present, but it is still possible to identify the three sub-intervals. The interesting feature for this string is that the third pass does not contain a sub-interval of common intersection since $k = 3$ and $(N + 1) = 9$ share a factor. Correspondingly, in Figure 3.16, at the third intersection of the lines A' and Z' there are two other lines present as well, and the slopes of these lines are such that immediately to the left or right of this point, the lines come between those of A' and Z' .

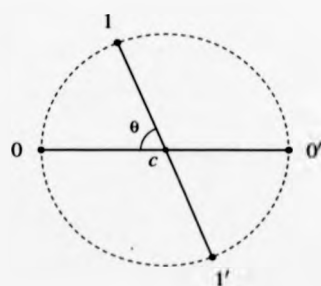
Clearly, the graphical method is not practical beyond the very shortest strings, and we now present a general process for finding the sub-intervals of common intersection. We require the following two results, describing some additional geometric properties of the rhombus configuration.

Lemma 3.9

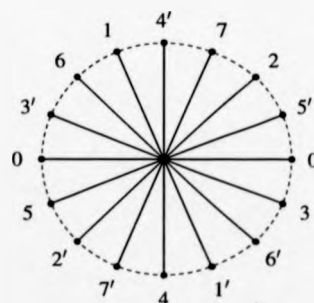
When $\theta = \theta_B$ or $\theta = \theta_E$, the points of contact of the central rhombi are equally spaced around the central circle, and θ is an integral multiple of the angle between two neighbouring points of contact.

Proof

We consider the collection of points of contact of the central rhombi, starting with central rhombus 1 which has two pairs of diametrically opposite points of contact. We join the two points in each pair to obtain two diameters of the central circle inclined at an angle θ , as shown in Figure 3.17(a). We do likewise for all subsequent central rhombi. Each introduces a further diameter (only one, because its other diameter coincides with a diameter from the previous central rhombus), and that diameter is inclined at an angle θ to the previous diameter, so for all $(N - 1)$ central rhombi there are N such diameters. Figure 3.17(b) illustrates this where the diameters are identified by the label r written next to the point of contact whose position angle is $\frac{3\pi}{2} + r\theta$. Note that, as in the figure, the diameters arising from the later rhombi can come between ones from earlier rhombi, so the angle between neighbouring points of contact need *not* be θ .



(a) Diameters from central rhombus 1



(b) Diameters from central rhombi 1, 2, ..., 7

Figure 3.17: The diameters obtained by joining opposite points of contact of the central rhombi.

We now consider adding one extra diameter to the collection, inclined at a further angle θ to the diameter contributed by central rhombus $(N-1)$. One of the two points of contact on this new diameter must coincide with $p_{2'}$ (because $p_{2'}$ is one of the two points of contact of central rhombus Z not shared with rhombus $(N-1)$). Since θ equals θ_B or θ_E , $p_{2'}$ coincides with either p_- or p_+ , and hence with one of the points of contact of a central rhombus. Consequently the new diameter falls exactly on top of one of the N diameters already present, the r^{th} diameter, say. Thus the angles of the new diameter and diameter r are the same (modulo π), that is

$$\frac{3\pi}{2} + N\theta = \frac{3\pi}{2} + r\theta + l\pi, \quad (3.30)$$

so $\theta = \frac{2\pi l}{2(N-r)}$, a rational multiple of 2π . We may thus write $\theta = 2\pi \frac{p}{q}$ for coprime integers p and q .

When p and q are coprime, the sequence $0, p, 2p, \dots, (q-1)p$ (where each is taken modulo q) consists of the integers $0, 1, \dots, q-1$ taken in some order. So if we select a point on the central circle and record its orbit under $(q-1)$ successive rotations by $\theta = 2\pi \frac{p}{q}$, we obtain q equally spaced points around the circle, and the q^{th} rotation returns to the original point. We do exactly this for the point of contact labelled 0 . Its first $(q-1)$ rotations by θ , which are the points of contact labelled $1, 2, \dots, (q-1)$, are therefore equally spaced around the central circle (in some order), the point q is coincident with 0 , and subsequent points $(q+1), (q+2), \dots, (N-1)$ each coincide with one of the points $0, 1, \dots, (q-1)$.

Finally, we consider the points of contact at the opposite ends of the diameters, which we will refer to as $0', 1', \dots, (N-1)'$. Since each point r' is conjugate to r under rotation by π about the centre of the circle, these N points are also coincident with q equally spaced points around the central circle. We take the point $0'$ which is either coincident with one of the points $1, 2, \dots, (q-1)$, or it is not. In the former case all the points $0', 1', \dots, (N-1)'$ are coincident with one of the points $0, 1, 2, \dots, (q-1)$, so there are q equally spaced points around the central circle. In the latter case none of the points $0', 1', \dots, (q-1)'$ coincide with a point $0, 1, \dots, (q-1)$, so there are $2q$ points around the central circle and,

because of the rotation taking r' to r , each point r' falls midway between two neighbouring points from the set $1, 2, \dots, (q-1)$.

Thus we have shown that the points of contact of the central rhombi are equally spaced around the central circle, and that $\theta = 2\pi \frac{p}{q}$ where either q or $2q$ is the number of points of contact. Hence either $\frac{2\pi}{q}$ or $\frac{\pi}{q}$ is the angle between neighbouring points and in both cases θ is an integral multiple of this angle. \square

Lemma 3.10

The number of distinct points of contact of the central rhombi with the central circle is at least 4 and at most $2N$.

Proof

Central rhombus 1, which is present in the rhombus configuration for any string $+0 \dots 0+$, has 4 distinct points of contact so the central polygon has at least 4 sides. Any of the subsequent central rhombi shares two points of contact with its predecessor, and the position angles of the remaining two points differ by π , so the next rhombus contributes at most two extra points of contact to the total (it may not contribute any, if the two extra points are coincident with those from an earlier rhombus). The rhombus configuration for a string of length N has $(N-1)$ central rhombi, so the maximum number of distinct points of contact is $4 + 2(N-2) = 2N$. \square

Remark : Both of these results have an alternative and significant interpretation in terms of the geometry of the central polygon. As a consequence of Lemma 3.9, the central polygon is regular when θ is θ_B or θ_E . This follows because the collection of points of contact of the central rhombi is invariant under a rotation by the angle determined by two adjacent points, and consequently the central polygon, viewed as the intersection of half-planes defined by the tangents at each of these points of contact, must also be invariant. We can see all of this occurring in plots of the configurations : for example, the central polygon for the string

$+00+$ is a regular octagon when $\theta = \frac{3\pi}{4}$ (the start of the sub-interval of common intersection for the second pass). This is the configuration which appeared as Figure 3.7(m), and it has been redrawn in Figure 3.18 to highlight the central polygon. Our second result, Lemma 3.10, is equivalent to the statement that the central polygon has at least 4 and at most $2N$ sides, because the number of sides of the central polygon equals the number of distinct points of contact. Taking again, by way of example, the string $+00+$, the central polygon has 4, 6 or 8 sides, and instances of all three are present amongst the configurations in Figure 3.7: the central polygon has 4 sides in Figure 3.7(i), 6 sides in Figure 3.7(l) and 8 sides in each of the other plots.

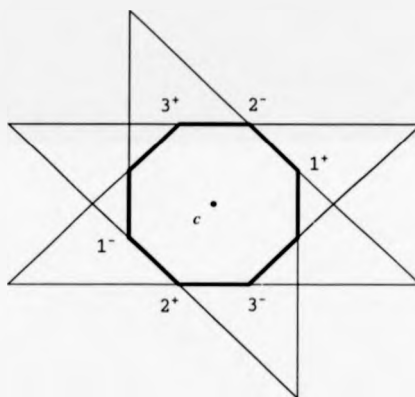


Figure 3.18: To illustrate that the central polygon is regular when $\theta = \theta_B$ or θ_E ; the plot shows the rhombus configuration for $+00+$ at $\theta = \frac{3\pi}{4}$.

We commence the presentation of the general method by locating θ_B , the start of the interval of common intersection within the k^{th} pass. As we saw in the proof of Proposition 3.8, the points of contact of the eccentric rhombi $p_{A'}$ and $p_{Z'}$ are coincident with p_- and p_+ respectively. The position angles of $p_{A'}$ and $p_{Z'}$ are (after reducing modulo 2π),

$$\angle vcp_{A'} = \frac{3\pi}{2} - \theta_B \quad (3.31)$$

$$\angle vcp_{Z'} = \frac{3\pi}{2} + N\theta_B - 2k\pi, \quad (3.32)$$

so the angle $\beta = \angle p_A \cdot c p_Z$ between p_A and p_Z is

$$\beta = 2k\pi - (N+1)\theta_B. \quad (3.33)$$

Now β is also the angle between p_- and p_+ which are neighbouring points of contact, so by Proposition 3.9,

$$\beta = \frac{2\pi}{s} \quad (3.34)$$

where s is the number of distinct points of contact of the central rhombi with the central circle. But from Proposition 3.10, s assumes one of the values 4, 6, 8, ..., $2N-2$, $2N$, and correspondingly there are only $(N-1)$ possible values for β . Rearranging equation (3.33),

$$\theta_B = \frac{2k\pi - \beta}{N+1} = \frac{2\pi}{N+1} \frac{ks-1}{s}, \quad (3.35)$$

which allows us to determine θ_B up to one of a set of $(N-1)$ possible values.

As an illustration we will list these possible values for θ_B for the first pass in the case of the string $+00+$. Here $N=4$ and $k=1$, so s will be one of 4, 6 or 8. The following table lists the values of β and θ_B corresponding to these three possibilities.

| s | β | θ_B |
|-----|-----------------|-------------------|
| 4 | $\frac{\pi}{2}$ | $\frac{3\pi}{10}$ |
| 6 | $\frac{\pi}{3}$ | $\frac{\pi}{3}$ |
| 8 | $\frac{\pi}{4}$ | $\frac{7\pi}{20}$ |

In this case we remark that the correct value of θ_B is $\frac{\pi}{3}$ as we saw in the plots of the configurations for the string $+00+$ in Section 3.5.

To select the correct choice from amongst the possibilities for s in the general case, we use the additional information from Proposition 3.9 that θ_B is an integral multiple of β . Writing m for this multiple, we have $\theta_B = \frac{2m\pi}{s}$ and after equating this with the expression

for θ_B in (3.35) and simplifying we obtain

$$ks = 1 + (N+1)m \quad (3.36)$$

so that the correct choice for s satisfies the linear congruence

$$ks \equiv 1 \pmod{N+1}. \quad (3.37)$$

Moreover this congruence determines s uniquely. Since k and $(N+1)$ are coprime, the theory of linear congruences asserts that there is a unique value of s in the range $0, 1, \dots, N$ satisfying (3.37). But clearly $s = 0$ cannot satisfy (3.37), and likewise $s = 2$ is not possible, for that would imply $k = \frac{1}{2}(N+1)m + \frac{1}{2}$ and there is no choice for the integer m consistent with $1 \leq k \leq \frac{N}{2}$, so the unique solution of congruence (3.37) is among the $(N-1)$ integers $1, 3, 4, 5, \dots, N$. Since s is necessarily even, if we find the solution to the congruence (3.37) is odd we add $(N+1)$, thereby obtaining a unique value for s from $4, 6, \dots, 2N$. Formula (3.35) then gives θ_B .

The upper end-point, θ_E , of the sub-interval of common intersection may be found by similar techniques. In this case $p_{A'}$ is coincident with p_- and $p_{Z'}$ is coincident with p_+ , so the angle β is $\angle vcp_{Z'} - \angle vcp_{A'} = (N+1)\theta_E - 2k\pi$. Using the same argument as above but with this new formula for β , it follows that s (again the number of distinct points of contact of the central rhombi at $\theta = \theta_E$) must satisfy the congruence

$$ks \equiv -1 \pmod{N+1} \quad (3.38)$$

and that θ_E is related to s by the formula

$$\theta_E = \frac{2\pi}{N+1} \frac{ks+1}{s}. \quad (3.39)$$

Again the congruence (3.38) determines s uniquely, but there is a slight complication here because unlike previously we cannot now eliminate the solution $s = 2$. But it follows from (3.38) that $s = 2$ if and only if $k = \frac{N}{2}$, so the value $s = 2$ can only arise on the final pass, and moreover always does so. For all the other passes, where $k = 1, 2, \dots, \frac{N}{2} - 1$, there is

no difficulty and the congruence (3.38) gives a unique value of s in the range $4, 6, \dots, 2N$ (after adding $(N+1)$ if necessary). For the final pass, where $k = \frac{N}{2}$, there is no s in the range $4, 6, \dots, 2N$ which satisfies (3.38), so we cannot identify a value θ_E for the upper end-point of the sub-interval of common intersection. The explanation here is that the final pass is $\left[\frac{(2N-1)\pi}{N}, \pi \right)$, and that a common intersection of rhombi $A', 1, 2, \dots, (N-1), Z'$ persists for θ approaching π . Note in fact that the formula (3.39) correctly gives $\theta_E = \pi$ when we substitute $s = 2$.

We are now in a position to complete the proof of Proposition 3.1, but to do so we must take account of the "missing boundaries" of each rhombus, in order that we can apply Proposition 3.2 to decide when the string appears as a substring of some admissible sequence. The point being that Proposition 3.2 applies to the set S rather than S' , or equivalently for our treatment, \bar{S} rather than \bar{S}' . The final step is not difficult, because we have shown (in Propositions 3.6 and 3.8) that the possibilities for the intersection set \bar{S}' are

- (a) \bar{S}' is empty,
- (b) \bar{S}' is a line segment or a single point,
- (c) \bar{S}' has non-empty interior.

Possibilities (a) and (c) present no difficulty because if \bar{S}' is empty then so is \bar{S} , and if \bar{S}' has non-empty interior then \bar{S} , obtained by removing a finite number of line segments, also has non-empty interior. In the remaining case (b), when \bar{S}' is a line segment or a single point, we note from the proofs of Propositions 3.6 and 3.8 that all the points in \bar{S}' lie on the edge of eccentric rhombus A' which is tangent to the central circle. But rhombus A' is simply $\mathcal{R} = T^{-1}(I^2)$, and the edge tangent to the central circle is edge cd (in the labelling of \mathcal{R} used in Section 3.3), and this is one of the edges which must be removed to obtain $T^{-1}(I^2)$. Hence when \bar{S}' is a line segment or a single point, then \bar{S} is empty.

We have seen in the proof of Propositions 3.6 and 3.8 that possibility (b) arises at the isolated value $\theta = \frac{2k\pi}{N+1}$ when N is odd or when N is even and $\text{hcf}(k, (N+1)) > 1$, and at the end-points θ_B and θ_E of the sub-intervals when N is even and $\text{hcf}(k, (N+1)) = 1$. Thus the set \bar{S} is :

- empty for all $0 < \theta < \pi$ when N is odd,
- non-empty for the open intervals (θ_B, θ_E) and empty outside of these intervals when N is even.

Our statement of Proposition 3.1 is now proved by invoking Proposition 3.2.

3.9 Application to other strings having regular structure

The methods employed in Sections 3.4 to 3.8, to discover the dependence of the common intersection of the rhombus configuration on the parameter θ , were devised with the specific class of strings $+0 \dots 0+$ in mind. However the ideas that underlie these methods do have wider application, and in this section we show how it is possible to determine when certain other classes of string appear as substrings of admissible sequences.

The section is divided into five parts. We begin, in 3.9.1, by establishing three symmetry results which show that if we make some simple modifications to a string $s_0 s_1 \dots s_{N-1}$ we obtain rhombus configurations which are closely related to those of the original string. These allow us to transfer our knowledge of the rhombus configurations for strings $+0 \dots 0+$ to those for other types of string.

In 3.9.2 and 3.9.3 we consider strings whose configurations involve only rhombi which we have already worked with in the configurations for the strings $+0 \dots 0+$. In each such case we can appeal directly to our earlier results in order to determine how the existence of a region of common intersection depends upon θ . In 3.9.4 we move beyond these easy cases, and there show how to modify our earlier arguments so as to cope with strings $-0 \dots 0+$.

Finally in 3.9.5, we consider a class of strings having an entirely different form, namely $0+-+ \dots +0$ and $0+-+ \dots -0$ (i.e. in which there is a run of alternating $+$ and $-$ digits between two 0 's) and we note that their rhombus configurations, although substantially different from those for the strings $+0 \dots 0+$, nevertheless share many of the features that permitted our previous analysis. Although we do not pursue the investigation here, it does

seem likely that with some modification, our methods will be effective for deciding when this significant type of string can appear as a substring of some admissible sequence.

3.9.1 Symmetry results for the rhombus configurations

The results here are of the same style as the symmetry results of Section 2.4, but are unrelated, because the results in Section 2.4 were strictly for periodic sequences. We use them in a similar way, enabling us to transfer results for the common intersection from one type of string to other types. In the statements below we employ a notation introduced in Chapter 2, where \bar{s}_i denotes the negated digit $-s_i$.

Proposition 3.11 (Negation of Strings)

The rhombus configuration for the string $s_0 s_1 \dots s_{N-1}$ at the parameter value θ has non-empty intersection if and only if the rhombus configuration for the string $\bar{s}_0 \bar{s}_1 \dots \bar{s}_{N-1}$ does.

Proof

Let A_0, A_1, \dots, A_N be the rhombi in the configuration for the string $s_0 s_1 \dots s_{N-1}$, defined as $A_r = \bar{\mathbf{w}}_r(\mathcal{R})$, and let B_0, B_1, \dots, B_N be the corresponding rhombi for the string $\bar{s}_0 \bar{s}_1 \dots \bar{s}_{N-1}$. Noting that $\mathbf{R}_\pi \mathcal{R} = \mathcal{R}$ and $\mathbf{R}_\pi \mathbf{T}^{-1} \mathbf{b} = -\mathbf{T}^{-1} \mathbf{b}$, and referring to (3.17) we have

$$\begin{aligned}
 \mathbf{R}_\pi B_r &= \mathbf{R}_{\pi+r\theta} \mathcal{R} + (\bar{s}_{N-r} \mathbf{R}_{\pi+(r-1)\theta} + \bar{s}_{N-r+1} \mathbf{R}_{\pi+(r-2)\theta} + \dots + \bar{s}_{N-1} \mathbf{R}_\pi) \mathbf{T}^{-1} \mathbf{b} \\
 &= \mathbf{R}_\theta \mathbf{R}_\pi \mathcal{R} - (s_{N-r} \mathbf{R}_{(r-1)\theta} + s_{N-r+1} \mathbf{R}_{(r-2)\theta} + \dots + s_{N-1} \mathbf{I}) \mathbf{R}_\pi \mathbf{T}^{-1} \mathbf{b} \\
 &= \mathbf{R}_\theta \mathcal{R} + (s_{N-r} \mathbf{R}_{(r-1)\theta} + s_{N-r+1} \mathbf{R}_{(r-2)\theta} + \dots + s_{N-1} \mathbf{I}) \mathbf{T}^{-1} \mathbf{b} \quad (3.40) \\
 &= A_r,
 \end{aligned}$$

so each rhombus A_r is a rotation of B_r by π about the origin. Consequently the common intersection of A_0, A_1, \dots, A_N is non-empty precisely when the common intersection of B_0, B_1, \dots, B_N is non-empty. \square

Proposition 3.12 (Reversal of Strings)

The rhombus configuration for the string $s_0 s_1 \dots s_{N-1}$ at the parameter value θ has non-empty intersection if and only if the rhombus configuration for its reversal $s_{N-1} \dots s_1 s_0$ does.

Proof

With A_0, A_1, \dots, A_N denoting the rhombi for the string $s_0 s_1 \dots s_{N-1}$, and B_0, B_1, \dots, B_N the rhombi for its reversal, we show that there is a transformation \mathbf{H} (a composition of a translation, a rotation and a reflection), so that $A_r = \mathbf{H}(B_r)$. We define \mathbf{H} by

$$\mathbf{H}(\mathbf{x}) = \mathbf{M} \mathbf{R}_{-(N-1)\theta} \{ \mathbf{x} - (s_{N-1} \mathbf{R}_{(N-1)\theta} + s_{N-2} \mathbf{R}_{(N-2)\theta} + \dots + s_0 \mathbf{I}) \mathbf{T}^{-1} \mathbf{b} \} \quad (3.41)$$

where $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the matrix implementing reflection in the horizontal axis. We note that

$$\mathbf{M} \mathbf{R}_{-\theta} = \mathbf{R}_\theta \mathbf{M}, \quad \mathbf{M} \mathcal{R} = \mathbf{R}_\theta \mathcal{R}, \quad \mathbf{M} \mathbf{T}^{-1} \mathbf{b} = -\mathbf{T}^{-1} \mathbf{b} \quad (3.42)$$

and that the rhombi A_r and B_{N-r} are given by

$$A_r = \mathbf{R}_{r\theta} \mathcal{R} + (s_{N-r} \mathbf{R}_{(r-1)\theta} + s_{N-(r-1)} \mathbf{R}_{(r-2)\theta} + \dots + s_{N-1} \mathbf{I}) \mathbf{T}^{-1} \mathbf{b}, \quad (3.43)$$

$$B_{N-r} = \mathbf{R}_{(N-r)\theta} \mathcal{R} + (s_{N-r-1} \mathbf{R}_{(N-r-1)\theta} + s_{N-r-2} \mathbf{R}_{(N-r-2)\theta} + \dots + s_0 \mathbf{I}) \mathbf{T}^{-1} \mathbf{b}. \quad (3.44)$$

Consequently,

$$\begin{aligned} \mathbf{H}(B_{N-r}) &= \mathbf{M} \mathbf{R}_{-(N-1)\theta} \{ \mathbf{R}_{(N-r)\theta} \mathcal{R} - (s_{N-1} \mathbf{R}_{(N-1)\theta} + s_{N-2} \mathbf{R}_{(N-2)\theta} + \dots \\ &\quad + s_{N-r} \mathbf{R}_{(N-r)\theta}) \mathbf{T}^{-1} \mathbf{b} \} \\ &= \mathbf{M} \mathbf{R}_{-(r-1)\theta} \mathcal{R} - \mathbf{M} (s_{N-1} \mathbf{I} + s_{N-2} \mathbf{R}_{-\theta} + \dots + s_{N-r} \mathbf{R}_{-(r-1)\theta}) \mathbf{T}^{-1} \mathbf{b} \\ &= \mathbf{R}_{r\theta} \mathcal{R} + (s_{N-1} \mathbf{I} + s_{N-2} \mathbf{R}_\theta + \dots + s_{N-r} \mathbf{R}_{(r-1)\theta}) \mathbf{T}^{-1} \mathbf{b} \\ &= A_r. \end{aligned} \quad (3.45)$$

□

Proposition 3.13 (Negation of Alternate Digits)

The rhombus configuration for the string $s_0 s_1 \dots s_{N-1}$ at the parameter value θ has non-empty intersection if and only if the rhombus configuration for the string $s_0 \bar{s}_1 \dots \bar{s}_{N-2} s_{N-1}$ (N odd), or $s_0 \bar{s}_1 \dots s_{N-2} \bar{s}_{N-1}$ (N even), has non-empty intersection at the parameter value $\pi - \theta$.

Proof

We let A_0, A_1, \dots, A_N be the rhombi for the string $s_0 s_1 \dots s_{N-1}$ at parameter value θ , and B_0, B_1, \dots, B_N be the rhombi for the string $s_0 \bar{s}_1 \dots \bar{s}_{N-2} s_{N-1}$ (for N odd) or $s_0 \bar{s}_1 \dots s_{N-2} \bar{s}_{N-1}$ (for N even) at parameter value $\pi - \theta$. As in the proof of Proposition 3.12, \mathbf{M} is the matrix implementing reflection in the horizontal axis and, since we need to distinguish between the rhombus \mathcal{R} and the displacement $\mathbf{T}^{-1}\mathbf{b}$ at the parameter values θ and $\pi - \theta$, for the latter value we will write $\mathcal{R}|_{\pi-\theta}$ and $\mathbf{T}^{-1}\mathbf{b}|_{\pi-\theta}$. For N even we have

$$\begin{aligned}
 \mathbf{M}B_r &= \mathbf{M} \{ \mathbf{R}_{r(\pi-\theta)} \mathcal{R}|_{\pi-\theta} + ((-1)^{r s_{N-r}} \mathbf{R}_{(r-1)(\pi-\theta)} + \\
 &\quad (-1)^{r-1 s_{N-(r-1)}} \mathbf{R}_{(r-2)(\pi-\theta)} + \dots + s_{N-2} \mathbf{R}_{\pi-\theta} - s_{N-1} \mathbf{I}) \mathbf{T}^{-1}\mathbf{b}|_{\pi-\theta} \} \\
 &= \mathbf{R}_{r(\theta-\pi)} \mathbf{M} \mathcal{R}|_{\pi-\theta} - ((-1)^{r s_{N-r}} \mathbf{R}_{(r-1)\theta} \mathbf{R}_{-(r-1)\pi} + \\
 &\quad (-1)^{r-1 s_{N-(r-1)}} \mathbf{R}_{(r-2)\theta} \mathbf{R}_{-(r-2)\pi} + \dots + s_{N-2} \mathbf{R}_\theta \mathbf{R}_{-\pi} - s_{N-1} \mathbf{I}) \mathbf{T}^{-1}\mathbf{b} \\
 &= \mathbf{R}_r \mathbf{R}_{-\pi} \mathcal{R} + (s_{N-r} \mathbf{R}_{(r-1)\theta} + s_{N-(r-1)} \mathbf{R}_{(r-2)\theta} + \dots + s_{N-2} \mathbf{R}_\theta + s_{N-1} \mathbf{I}) \mathbf{T}^{-1}\mathbf{b} \\
 &= A_r,
 \end{aligned} \tag{3.46}$$

because $\mathbf{M} \mathcal{R}|_{\pi-\theta} = \mathcal{R}$, $\mathbf{T}^{-1}\mathbf{b}|_{\pi-\theta} = \mathbf{T}^{-1}\mathbf{b}$, and using the expression (3.43) for A_r which appears in the proof of Proposition 3.12. For N odd we have

$$\begin{aligned}
 \mathbf{R}_\pi \mathbf{M}B_r &= \mathbf{R}_\pi \mathbf{M} \{ \mathbf{R}_{r(\pi-\theta)} \mathcal{R}|_{\pi-\theta} + ((-1)^{r-1 s_{N-r}} \mathbf{R}_{(r-1)(\pi-\theta)} + \\
 &\quad (-1)^{r-2 s_{N-(r-1)}} \mathbf{R}_{(r-2)(\pi-\theta)} + \dots - s_{N-2} \mathbf{R}_{\pi-\theta} + s_{N-1} \mathbf{I}) \mathbf{T}^{-1}\mathbf{b}|_{\pi-\theta} \} \\
 &= \mathbf{R}_{r\theta} \mathcal{R} - \mathbf{R}_\pi ((-1)^{r-1 s_{N-r}} \mathbf{R}_{(r-1)(\theta-\pi)} + \\
 &\quad (-1)^{r-2 s_{N-(r-1)}} \mathbf{R}_{(r-2)(\theta-\pi)} + \dots - s_{N-2} \mathbf{R}_{\theta-\pi} + s_{N-1} \mathbf{I}) \mathbf{T}^{-1}\mathbf{b} \\
 &= \mathbf{R}_{r\theta} \mathcal{R} - \mathbf{R}_\pi (s_{N-r} \mathbf{R}_{(r-1)\theta} + s_{N-(r-1)} \mathbf{R}_{(r-2)\theta} + \dots + s_{N-2} \mathbf{R}_\theta + s_{N-1} \mathbf{I}) \mathbf{T}^{-1}\mathbf{b} \\
 &= A_r.
 \end{aligned} \tag{3.47}$$

□

Note that there is no equivalent of the "multiple copies" result for periodic sequences (Proposition 2.4, part 4) in the case of strings. The single digit string $+$ provides a simple counter example. The rhombus configuration for this string consists of only two rhombi, these are eccentric rhombus A' and central rhombus 1 from our definitions in Section 3.4, and have a common intersection for all $0 < \theta < \pi$ (this is easily checked by direct calculation of the triangular region of intersection; it has positive area except at $\theta = \frac{\pi}{2}$, where the intersection only occurs along the common edge between the two rhombi). By contrast, when the string $+$ is doubled, we obtain $++$ which is the case $N = 2$ of the family $+0 \dots 0+$, and our methods in Sections 3.7 and 3.8 show that there is a common intersection only for $\frac{\pi}{2} \leq \theta < \pi$.

3.9.2 Strings of the form $0 \dots 0$

The configuration for a string $0 \dots 0$ of length N consists of $(N + 1)$ copies of the rhombus \mathcal{R} , each centred at the origin and inclined at an angle θ to their predecessor. The unit circle centred at the origin is common to all the rhombi, and consequently the string appears in some admissible sequence for all $0 < \theta < \pi$. (Of course, from the point of view of the dynamical system, the origin in I^2 always gives rise to the sequence every digit of which is 0 .)

3.9.3 Strings $+0 \dots 0$, $0 \dots 0+$, $-0 \dots 0$ and $0 \dots 0-$

We begin with the observation that if we remove rhombus Z' from the configuration for a string $+0 \dots 0+$ of length N , we obtain the rhombus configuration for a string $0 \dots 0+$ of length $(N - 1)$. Thus we are interested in the dependence on θ of the common intersection of rhombi A' , 1, 2, ..., $(N-1)$ as defined in Section 3.4. We may immediately deduce that this collection of rhombi has a non-empty intersection *regardless* of the value of θ , because the point $p_{A'}$ (belonging to A') lies on the central circle (contained in each of the central rhombi). From our work on the configurations for strings $+0 \dots 0+$ we know the intersection between A' and the $(N - 1)$ central rhombi has non-empty interior, apart from the exceptional cases when $p_{A'}$ coincides with a point of contact of one of the central

rhombi. This latter case occurs when the position angle of $p_{A'}$, namely $\frac{3\pi}{2} - \theta$ differs by an integer multiple of 2π from the position angle of one of the points of contact of the central rhombi, namely $\frac{\pi}{2} + k\theta$ and $\frac{3\pi}{2} + k\theta$ for $k = 0, 1, \dots, (N-1)$.

Thus for a string $0 \dots 0+$ of length N , the common intersection of the rhombus configuration has non-empty interior except at the isolated values

$$\theta = \frac{l}{k+1}\pi \quad \text{for } k = 1, 2, \dots, N \quad \text{and} \quad 1 \leq l \leq k, \quad (3.48)$$

and at these values of θ the common intersection occurs along a line segment, part of the edge of rhombus A' tangent to the central circle. We may now argue as in Section 3.8 that this edge is removed to obtain the intersection needed for Proposition 3.2. Consequently the string $0 \dots 0+$ of length N appears within some admissible sequence for all $0 < \theta < \pi$, *except* at the finite set of isolated values of θ listed in (3.48).

Using the symmetry results, Propositions 3.11 and 3.12 above, we deduce that the rhombus configurations for the related strings $+0 \dots 0$, $0 \dots 0-$ and $-0 \dots 0$ have non-empty intersection for all $0 < \theta < \pi$, and this intersection has non-empty interior except at the values (3.48). Again, before we apply Proposition 3.2 to learn when the strings appear within some admissible sequence, we need to examine how the deletion of the forbidden edges affects the intersection, and the three types of string $+0 \dots 0$, $0 \dots 0-$ and $-0 \dots 0$ differ in this. We briefly summarise the situation for each of these types of string.

- (i) For the string $+0 \dots 0$ the configurations where the intersection is along a line segment involve one of the deleted edges of the eccentric rhombus, so this string appears within an admissible sequence for all $0 < \theta < \pi$ except at the values (3.48).
- (ii) In the cases of the other two strings, $-0 \dots 0$ and $0 \dots 0-$, the intersections along a line segment always involve a side of the eccentric rhombus which is not removed. Thus at some of the isolated values (3.48) it is possible that one of these strings can appear within an admissible sequence. We could work out precisely for which of the values (3.48) this occurs, and the task would be to check for each value (3.48) whether the point of contact of any central rhombus coinciding with $p_{A'}$ lies on a side of the

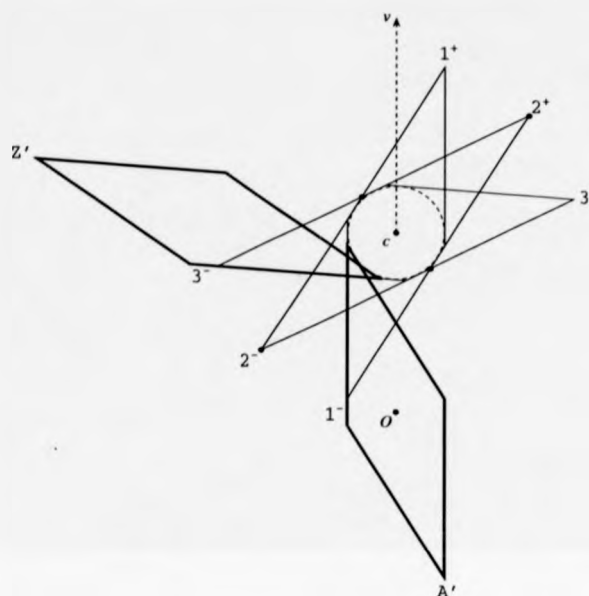
central rhombus which is not removed. We choose not to pursue here the details of this routine calculation.

3.9.4 Strings $-0\dots 0+$, $+0\dots 0-$ and $-0\dots 0-$

When N is even, we may apply Propositions 3.11 and 3.13 to a length N string $+0\dots 0+$ and deduce that, corresponding to every interval $[\theta_B, \theta_E]$ (or isolated value θ) where the rhombus configuration for the string $+0\dots 0+$ has non-empty intersection, there is an interval $[\pi - \theta_E, \pi - \theta_B]$ (or isolated value $\pi - \theta$) where the rhombus configuration for the string $-0\dots 0+$ has non-empty intersection. By contrast, the case when N is odd is a genuinely new situation : negating alternate digits of $+0\dots 0+$ does not now produce $-0\dots 0+$, and consequently the rhombus configuration for strings $-0\dots 0+$ of odd length cannot be deduced from the configuration for $+0\dots 0+$ by any of the symmetry results above.

It is possible, however, independently to make progress with the odd length strings $-0\dots 0+$ because the rhombus configuration for a string $-0\dots 0+$ is closely related to the configuration for the string $+0\dots 0+$, the two strings matching in all but the first digit. The rhombus configuration for a string $-0\dots 0+$ of length N (irrespective of whether N is odd or even) consists of two eccentric rhombi and $(N - 1)$ central rhombi, just as in our previous work. Moreover, eccentric rhombus A' (defined as $\bar{w}_0(\mathcal{R})$) and all the central rhombi (which are $\bar{w}_1(\mathcal{R}), \bar{w}_2(\mathcal{R}), \dots, \bar{w}_{N-1}(\mathcal{R})$) are unchanged from their descriptions in Section 3.4. Only rhombus Z' ($= \bar{w}_N(\mathcal{R})$) is altered; as before it is a translated copy of central rhombus Z , but for the configuration corresponding to the string $-0\dots 0+$ the translation is $-\mathbf{R}_{(N-1)\theta} \mathbf{T}^{-1} \mathbf{b}$, i.e. the negative of what it was before. By way of example, Figure 3.19 shows the configuration for the string $-00+$; this figure should be compared with Figure 3.5 (where the plot is for the string $+00+$), both for the same parameter value $\theta = 0.5$. Rhombi A' , 1, 2 and 3 are the same between the two plots, only the location of Z' is changed.

Owing to the similarity between the rhombus configurations for the strings $-0\dots 0+$ and $+0\dots 0+$, we can adapt the argument presented through Sections 3.4 to 3.8 and determine

Figure 3.19: The rhombus configuration for the string $-00+$.

precisely when strings $-0 \dots 0+$ appear in some admissible sequence. The following result is the counterpart to Proposition 3.1. For its proof we list, briefly, the modifications which must be made to our earlier exposition.

Proposition 3.14 (Version of Proposition 3.1 for strings $-0 \dots 0+$)

The string $-0 \dots 0+$ is a substring of some admissible sequence if and only if θ belongs to one of the intervals $\left(\frac{\pi}{N+1} \frac{(2k-1)s_1-2}{s_1}, \frac{\pi}{N+1} \frac{(2k-1)s_2+2}{s_2} \right)$ where $1 \leq k \leq \left\lceil \frac{N}{2} \right\rceil$ and s_1, s_2 are the least positive even integers satisfying the congruences

$$(2k-1)s_1 \equiv 2 \pmod{2(N+1)} \quad \text{and} \quad (2k-1)s_2 \equiv -2 \pmod{2(N+1)}. \quad (3.49)$$

Proof

We have already described the differences between the rhombus configurations for the strings $+0 \dots 0+$ and $-0 \dots 0+$. Modulo the change in location of Z' , the material in Section

3.4 applies to the configurations for $-0 \dots 0+$; note though, that the position angle of $p_{z'}$, the point of contact of z' with the central circle, must be changed to $\frac{\pi}{2} + N\theta$.

The rhombus configuration for strings $-0 \dots 0+$ still has a reflection symmetry, but the axis is affected by the new location of z' , and we must redefine σ at right-angles to its previous orientation so that the new axis σ is specified by the angle $\angle_{vc\sigma} = \frac{1}{2}(N-1)\theta + \pi$. We saw in Section 3.5 that the central rhombi are symmetric in this axis (recall that there was a reflection in a second axis perpendicular to the original σ) so to establish Proposition 3.3 for strings $-0 \dots 0+$ we must additionally demonstrate that A' and Z' are conjugate under the new reflection. This follows because a reflection of A' in the new axis σ is equivalent to a reflection of A' in the old axis σ (giving the old rhombus Z') followed by a rotation of π about c . All that remains to show is that the new rhombus Z' is obtained from the old rhombus Z' by a rotation of π about c . This rotation is effected by the map $y \mapsto R_{\pi}(y - T^{-1}b) + T^{-1}b$, so that the old rhombus $Z' = z + R_{(N-1)\theta}T^{-1}b$ is rotated to

$$R_{\pi}(R_{N\theta}z + R_{(N-1)\theta}T^{-1}b) + T^{-1}b = z - R_{(N-1)\theta}T^{-1}b, \quad (3.50)$$

i.e. to the new rhombus Z' . The axis σ meets the central polygon either at a vertex or at the mid-point of a side; the latter case occurring at one of the values

$$k\theta + l\pi = \frac{1}{2}(N-1)\theta + \frac{\pi}{2} \quad (3.51)$$

where $k = 0, 1, \dots, (N-1)$ and l is any integer such that $0 < \theta < \pi$. (Unlike the previous situation σ does not always pass through the midpoint of a side when N is odd.) From (3.51) we can list these exceptional values as

$$\theta = \frac{2p-1}{q}\pi \quad \text{for} \quad \begin{cases} q = 2, 4, 6, \dots, (N-1), & 1 \leq p \leq \frac{1}{2}q & \text{when } N \text{ is odd} \\ q = 3, 5, 7, \dots, (N-1), & 1 \leq p \leq \frac{1}{2}(q-1) & \text{when } N \text{ is even} \end{cases}$$

The intervals of θ for which A' and Z' intersect (i.e. the passes) are changed. Parts (i) and (ii) of Proposition 3.5 hold in the present case, so the values of θ for which A' and Z'

intersect satisfy

$$\pi - \frac{1}{2}\theta \leq \frac{1}{2}(N-1)\theta + \pi + l\pi \leq \frac{3\pi}{2} - \frac{1}{2}\theta \quad (3.52)$$

for some integer l such that $0 < \theta < \pi$. After rearrangement we obtain $\lceil \frac{N}{2} \rceil$ intervals for θ where A' and Z' intersect; these passes are $\left[\frac{(2k-2)\pi}{N}, \frac{(2k-1)\pi}{N} \right]$, $k = 1, 2, \dots, \lceil \frac{N}{2} \rceil$. With these new values, part (iv) of Proposition 3.5 is readily established. We do not use Proposition 3.6, instead proceeding without separating the strings $-0 \dots 0+$ into cases N odd or N even. Proposition 3.7 transfers to the new configuration without modification, and Proposition 3.8 is now needed for both N odd and N even. The argument for the proof is not affected, but two details must be changed: the isolated value within the k^{th} pass at which $p_{A'}$ and $p_{Z'}$ are coincident becomes $\theta = \frac{(2k-1)\pi}{N+1}$, and the common intersection occurs for just this single value when $(2k-1)$ and $(N+1)$ share a common factor.

Lemmas 3.9 and 3.10 involve the central rhombi alone, so apply without change to strings $-0 \dots 0+$. In the argument to establish the end-points of the intervals of common intersection, the position angle of $p_{Z'}$ is different: $\angle vcp_{Z'} = \frac{\pi}{2} + N\theta_B - 2(k-1)\pi$ so $\beta = (2k-1)\pi - (N+1)\theta_B$ and we obtain $\theta_B = \frac{\pi}{N+1} \frac{(2k-1)s_1 - 2}{s_1}$ where s_1 satisfies the congruence $(2k-1)s_1 \equiv 2 \pmod{2(N+1)}$. Similarly for the upper end-point of the interval we get $\theta_E = \frac{\pi}{N+1} \frac{(2k-1)s_2 + 2}{s_2}$ where s_2 satisfies $(2k-1)s_2 \equiv -2 \pmod{2(N+1)}$. Because $(2k-1)$ and $(N+1)$ are coprime, as previously these congruences determine s_1 and s_2 uniquely: clearly the solutions for s_1 and s_2 are one of $0, 2, 4, \dots, 2N$ because $(2k-1)$ is odd and $2(N+1)$ is even; neither can be zero, and it is straightforward to show that $s_1 = 2$ only on the first pass (corresponding to an interval for θ with left end-point zero). The case of s_2 is a little more complicated: when N is even s_2 can never equal 2, and when N is odd $s_2 = 2$ only on the final pass (corresponding to an interval for θ with right end-point π).

Finally we note that in those instances when the common intersection is a line segment (i.e. at isolated values $\theta = \frac{(2k-1)\pi}{N+1}$ or at the end-points θ_E and θ_B of the sub-intervals), the line segment coincides with a deleted edge of rhombus A' , so as in Section 3.8, the string does not appear in any admissible sequences at one of these values. \square

| String Length (N) | String | Intervals for θ |
|--------------------------|-------------|---|
| 2 | -+ | $(0, \frac{\pi}{2})$ |
| 3 | -0+ | $(0, \frac{\pi}{3}), (\frac{2\pi}{3}, \pi)$ |
| 4 | -00+ | $(0, \frac{\pi}{4}), (\frac{\pi}{2}, \frac{2\pi}{3})$ |
| 5 | -000+ | $(0, \frac{\pi}{5}), (\frac{4\pi}{5}, \pi)$ |
| 6 | -0000+ | $(0, \frac{\pi}{6}), (\frac{2\pi}{5}, \frac{\pi}{2}), (\frac{2\pi}{3}, \frac{3\pi}{4})$ |
| 7 | -00000+ | $(0, \frac{\pi}{7}), (\frac{\pi}{3}, \frac{2\pi}{5}), (\frac{3\pi}{5}, \frac{2\pi}{3}), (\frac{6\pi}{7}, \pi)$ |
| 8 | -000000+ | $(0, \frac{\pi}{8}), (\frac{\pi}{2}, \frac{4\pi}{7}), (\frac{3\pi}{4}, \frac{4\pi}{5})$ |
| 9 | -0000000+ | $(0, \frac{\pi}{9}), (\frac{2\pi}{7}, \frac{\pi}{3}), (\frac{2\pi}{3}, \frac{5\pi}{7}), (\frac{8\pi}{9}, \pi)$ |
| 10 | -00000000+ | $(0, \frac{\pi}{10}), (\frac{\pi}{4}, \frac{2\pi}{7}), (\frac{4\pi}{9}, \frac{\pi}{2}), (\frac{5\pi}{8}, \frac{2\pi}{3}), (\frac{4\pi}{5}, \frac{5\pi}{6})$ |
| 11 | -000000000+ | $(0, \frac{\pi}{11}), (\frac{2\pi}{5}, \frac{3\pi}{7}), (\frac{4\pi}{7}, \frac{3\pi}{5}), (\frac{10\pi}{11}, \pi)$ |

Table 3.5: Values of θ for which strings $-0 \dots 0+$ appear within admissible sequences.

Table 3.5 lists the intervals for which short strings $-0 \dots 0+$ appear within some admissible sequence. Comparison with Table 3.1 reveals the correspondence mentioned at the start of this section, in the case when N is even, between an interval (θ_B, θ_E) for a string $+0 \dots 0+$ and the interval $(\pi - \theta_E, \pi - \theta_B)$ for the string $-0 \dots 0+$. We observe also the symmetric distribution of the intervals when N is odd : an interval (θ_B, θ_E) appears in Table 3.5 if and only if an interval $(\pi - \theta_E, \pi - \theta_B)$ does. We note that it is possible to prove these correspondences directly from the congruences (3.1) and (3.49), but since this is purely a task of algebraic manipulation we do not include it here.

Remark : Now that we have determined precisely the values of the parameter θ when the strings $+0 \dots 0+$ and $-0 \dots 0+$ can appear as substrings of admissible sequences, we return briefly to the earlier result of Galias & Ogorzałek (1992) and demonstrate how it fits into our more general framework. Recall that their result states that for a given value of θ , one of the two length N strings $+0 \dots 0+$ and $-0 \dots 0+$ cannot appear within any admissible sequence, specifically the former string when $\sin N\theta > 0$ and the latter when $\sin N\theta < 0$. Viewed in terms of our geometric approach, these trigonometric conditions correspond simply to the restriction that the eccentric rhombi A' and Z' do not intersect, as we show. For a string

$+0 \dots 0+$ of length N , we know from part (iv) of Proposition 3.5 that the condition for A' and Z' not to intersect is that $p_{Z'}$ should lie on the upper half of the central circle. This is equivalent to $\cos\left(\frac{3\pi}{2} + N\theta\right) > 0$ (since the position angle of $p_{Z'}$ is $\frac{3\pi}{2} + N\theta$ and we are measuring angles clockwise from the upward vertical), which simplifies to $\sin N\theta > 0$. The other case, concerning the strings $-0 \dots 0+$, follows in like manner. In terms of the notions that we have introduced, the result of Galias & Ogorzalek (1992) says that a string cannot appear within any admissible sequence unless θ belongs to one of the intervals we called a *pass* (see page 91). Within each pass, determining the *precise* interval (θ_B, θ_E) where the string can appear within some admissible sequence has needed a much more extensive analysis, the point being we are concerned with whether the intersection of all the rhombi is non-empty, as opposed to just the pair of rhombi A' and Z' .

Finally we mention the related strings $+0 \dots 0-$ and $-0 \dots 0-$. The values of θ for which their rhombus configurations have non-empty intersection are easily deduced from our results for the strings $-0 \dots 0+$ and $+0 \dots 0+$ via Proposition 3.11. When we describe the intervals for which the strings $+0 \dots 0-$ and $-0 \dots 0-$ can appear as substrings of admissible sequences, similar comments to those made in Section 3.9.3 apply concerning intersections along line segments involving deleted edges of the boundaries of the rhombi. Taking account of the particular edges of A' and Z' that are removed, it follows that Proposition 3.14 describes the values of θ precisely for strings $+0 \dots 0-$, but for strings $-0 \dots 0-$, we may have to include some single points $\theta = \frac{2k\pi}{N+1}$, along with the intervals specified in Proposition 3.1, and some of the intervals may include (one or both of) their end-points; again the precise details could be worked out, but are not really of interest to us.

3.9.5 Strings $0+-+ \dots +0$ and $0+-+ \dots -0$

The strings $0+-+ \dots +0$ and $0+-+ \dots -0$, consisting of a run of alternating $+$ and $-$ digits with zeros at either end, are substantially different in form to those we have considered above, and their rhombus configurations are quite unlike those for $+0 \dots 0+$ or $-0 \dots 0+$. Nevertheless, when we plot rhombus configurations for the strings $0+-+ \dots +0$ and study

their evolution with the parameter θ , we encounter again the features which were crucial in our argument for determining the common intersection for the strings $+0 \dots 0+$. As before we may separate the $(N+1)$ rhombi of the configuration into 2 "eccentric" rhombi A' and Z' (the first and last of the rhombi arising as $\bar{w}_0(\mathcal{R})$ and $\bar{w}_N(\mathcal{R})$), and the other $(N-1)$ "central" rhombi $1, 2, \dots, (N-1)$.

Studying examples of these configurations, two of which are presented in Figures 3.20 and 3.21, suggests that the following familiar features are present :

1. There is a "central" circle, situated within the rhombi $1, 2, \dots, (N-1)$, which is tangent to two (but not four) sides of each rhombus. As previously, "eccentric" rhombi A' and Z' lie outside of this circle, but each touch it at a single point. A difference is though, that here the circle varies in size, as well as location, as θ increases.
2. There is a reflection symmetry for each configuration, whose axis passes through the centre of the "central" circle.
3. A non-empty intersection of the rhombi $A', 1, \dots, (N-1), Z'$ occurs for an interval of θ , or at exceptional isolated values for which the common point of contact of A' and Z' is coincident with a point of contact of a "central" rhombus.
4. The "central" polygon is regular at values of θ corresponding to the start and end of an interval of common intersection.

The presence of these features points strongly to the possibility of adapting the methods of this chapter to describe in general the common intersections for a string $0+-+\dots+0$ or $0+-+\dots-0$ (and consequently also the strings $0--+\dots-0$, $0--+\dots+0$, $0++\dots++0$ and $0--\dots--0$). We will not attempt this here though; we envisage it would require a lengthy presentation of ideas similar to those already discussed.

For short strings the evolution of the configuration can be studied by computer; by this means we have been able to determine the values of θ that correspond to a common intersection for the string lengths $N = 3, 4, \dots, 11$. Table 3.6 lists the ranges of θ which we found : they are either intervals or single points, which is consistent with there being a

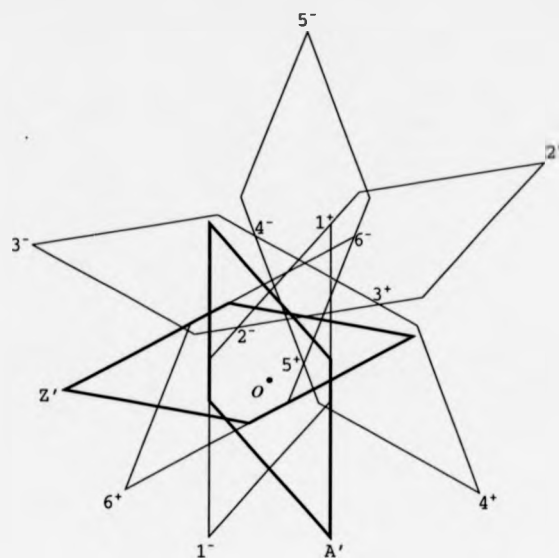


Figure 3.20: A typical plot of the rhombus configuration for the string $0+-+--+0$, when $\theta = 0.7$.

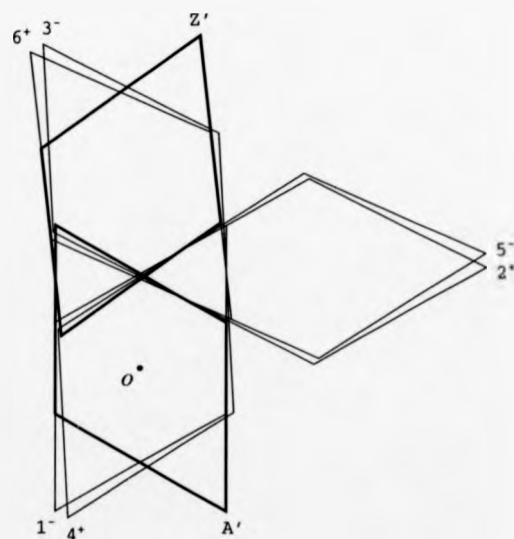


Figure 3.21: The configuration near $\theta = \frac{\pi}{3}$, where several of the sides of the various rhombi become co-linear.

| String Length (N) | String | Values of θ |
|--------------------------|------------|---|
| 3 | $0+0$ | $[\frac{\pi}{3}, \frac{2\pi}{3}]$ |
| 4 | $0+-0$ | $(0, \frac{\pi}{3}]$ |
| 5 | $0+--+0$ | $\frac{\pi}{3}$ |
| 6 | $0+---0$ | $(0, \frac{\pi}{5}], [\frac{\pi}{3}, \frac{\pi}{2}]$ |
| 7 | $0+----0$ | $[\frac{\pi}{5}, \frac{\pi}{3}]$ |
| 8 | $0+-----0$ | $(0, \frac{\pi}{7}], \frac{\pi}{3}$ |
| 9 | $0+-----0$ | $\frac{\pi}{5}, [\frac{\pi}{3}, \frac{3\pi}{7}]$ |
| 10 | $0+-----0$ | $(0, \frac{\pi}{9}], [\frac{\pi}{4}, \frac{\pi}{3}], [\frac{3\pi}{7}, \frac{\pi}{2}]$ |
| 11 | $0+-----0$ | $[\frac{\pi}{7}, \frac{\pi}{5}], \frac{\pi}{3}$ |

Table 3.6: Values of θ where there is a common intersection for the rhombus configuration for strings $0+--+ \dots +0$.

description in terms of congruences similar to our descriptions for the strings $+0 \dots 0+$ and $-0 \dots 0+$. We should mention that, according to Table 3.6, when N exceeds 3 a common intersection only occurs within the first half-range $0 < \theta \leq \frac{\pi}{2}$. This is entirely to be expected because when $N > 3$ the strings all contain the substring $+-$, and we know from Section 3.9.4 that $+-$ can only appear in an admissible sequence when $0 < \theta \leq \frac{\pi}{2}$.

Chapter Summary

- It is shown that finite strings of the form $+0 \dots 0+$ may appear within some admissible sequence for only a restricted set of values of the parameter θ . More specifically the string $+0 \dots 0+$ of length N is a substring of some admissible sequence if and only if θ belongs to one of the following intervals : for each k such that $1 \leq k \leq \lfloor \frac{N}{2} \rfloor$ determine, where possible, the least positive even integers s_1, s_2 satisfying $ks_1 \equiv 1 \pmod{N+1}$ and $ks_2 \equiv -1 \pmod{N+1}$, and an interval is $\left(\frac{2\pi}{N+1} \frac{ks_1-1}{s_1}, \frac{2\pi}{N+1} \frac{ks_2+1}{s_2} \right)$.
- The proof is geometric in character and develops from the condition that a string $s_0 s_1 \dots s_{N-1}$ appears within some admissible sequence if and only if the intersection of $(N+1)$ congruent rhombi $\bar{\mathbf{w}}_0(\mathbf{T}^{-1}(I^2)) \cap \bar{\mathbf{w}}_1(\mathbf{T}^{-1}(I^2)) \cap \dots \cap \bar{\mathbf{w}}_N(\mathbf{T}^{-1}(I^2))$, where $\bar{\mathbf{w}}_r(\mathbf{y}) = \mathbf{R}_{r\theta} \mathbf{y} + \sum_{i=0}^{r-1} s_{N-1-i} \mathbf{R}_{i\theta} \mathbf{T}^{-1} \mathbf{b}$, is non-empty. As the rhombus configuration evolves with increasing θ , the presence of a common intersection for the case of the string $+0 \dots 0+$ is linked to the relative locations of the points of contact of the sides of the rhombi which are tangent to a common inscribed circle.
- Related results are obtained for the strings $+0 \dots 0, -0 \dots 0, 0 \dots 0+, 0 \dots 0-, -0 \dots 0-, +0 \dots 0-$ and $-0 \dots 0+$. Precise descriptions are provided for the sets of θ values where each may appear within an admissible sequence, and these definitive statements encompass all previously published partial results.
- Corresponding rhombus configurations for strings $0+-+\dots 0$ are briefly examined; they share many of the geometric features significant for the treatment of the strings $+0 \dots 0+$ and so the strings $0+-+\dots 0$ are candidates for the further application of the techniques presented here.

Chapter 4

Search Strategies for the Computational Determination of Admissible Periodic Sequences

Of significant interest is an exploration of the periodic sequences that might be admissible for the nonlinear map which underlies this study. The criterion for admissibility, that derives from the association of a symbolic sequence with the succession of iterates of a point under the action of the nonlinear map, goes no way towards providing a general description of the form taken by the admissible periodic sequences, and there are very few theoretical results that detail their nature. Almost nothing seems to be known. This being so an alternative is to assemble empirical information with the hope of obtaining some insight through the study of the examples uncovered. Yet it is the nature of the task that useful information about the admissible periodic sequences is not readily obtained by computational means. Certainly the criterion for admissibility provides a simple test for deciding whether some given string generates an admissible periodic sequence, and it may be taken as the basis of a simple algorithm that tests in turn all the 3^N strings of length N which conceivably could generate an admissible periodic sequence. Of course the overhead of processing so many strings limits its viability, and the first reported computer studies, all of which used this approach, were restricted to string lengths $N \leq 25$, and to a single parameter value $a = 0.5$. In order to investigate sequences having longer period it is essential to effect a drastic reduction in the number of strings that need to be scrutinised. Galias and Ogorzałek (1992) proposed a means of achieving this and these authors

were able to extend the scope of the search to $N \leq 56$, but again only for $a = 0.5$. Unfortunately the scarcity of the admissible periodic sequences so uncovered means that the information obtained for these lengths still gives little insight into the nature of the sequences. Furthermore, the fact that the studies to date have concentrated on a single value for the parameter a implies that we have little comprehension of the interdependence between admissibility and parameter value.

Galias and Ogorzałek do not report the way their computations were conducted, but the limited scope of their results and omissions in their list of strings would suggest that their implementation was not very powerful. We present in this chapter a careful appraisal of what is required to study computationally the problem of seeking admissible periodic sequences, and give a detailed account of our own computational investigations. Inferences that we draw from the results form the basis of theoretical investigations in the succeeding chapters.

As part of the description of our computational strategy, fragments of computer code are included with the text, their purpose being to indicate the broad outline of the methods employed and also to aid the presentation of specific aspects. The fragments take the form of pseudo-code, loosely based on the C language, and the style of this code is informal. Although it does parallel the actual code used in our implementation, its primary aim is to be descriptive. In view of this we have suppressed language-specific details such as variable type information and function header descriptions. Frequently we will wish to refer to a section of code either explained elsewhere in the text or sufficiently simple not to warrant a separate explanation; in such cases we employ the symbol \blacktriangleright together with a line of summary text. For the benefit of readers not conversant with C syntax, a brief table appears below (Table 4.1) summarising those elements of the syntax that permit us to be concise and precise in the pseudo-code.

We comment here upon the reason that our computer programs are written in C. The choice of language was dictated by the need for the fastest possible speed of execution. With the aim being to uncover information about periodic sequences of as long period as possible, the foremost consideration at every stage of the implementation has been efficiency, both algorithmic and computational. In view of this we have avoided the use of more sophisticated data structures available in high-level languages, and of ready-made library functions, choosing instead to write our own taut and lean versions.

| | |
|--------|---|
| && | logical "and" |
| | logical "or" |
| ! | logical negation |
| == | test for equality (note that = indicates variable assignment) |
| i++ | increment <i>i</i> by 1 |
| i += 3 | increment <i>i</i> by 3 |
| a % b | the arithmetic remainder when <i>a</i> is divided by <i>b</i> ($= a \bmod b$) |

Table 4.1: Summary of the C syntax used in our pseudo-code.

4.1 Elimination of the strings which may not appear in any admissible sequence

The scheme we adopt to limit the proportion of the 3^N strings of length N that need to be examined, starts from an idea proposed in Galias & Ogorzałek (1992). Instead of commencing with a direct search for the strings which generate admissible periodic sequences, these authors suggest the insertion of a preliminary step, filtering on the basis of the alternative trial "Which strings can appear as a substring of *some* admissible sequence, for the map F at the given parameter value?". Clearly this latter property is a prerequisite for any string to generate an admissible periodic sequence, so we expect to be left with a smaller collection of strings to test after incorporating this initial step. However, the real advantage gained derives from the fact that the notion of a string $s_0s_1 \dots s_{N-1}$ "unable to appear in any admissible sequence" is *inherited* by any longer string which contains $s_0s_1 \dots s_{N-1}$ as a substring (which follows immediately from the definition of an admissible sequence given in Section 2.1.1). Thus we can devise a computational strategy that builds up from the shortest strings longer ones by inserting one digit at a time at the start. As soon as a string $s_0s_1 \dots s_{n-1}$ of length n is identified as one that cannot appear within any admissible sequence, we may reject it and thereafter never need to consider any of the 3^{N-n} longer strings which terminate with $s_0s_1 \dots s_{n-1}$. In this way we successfully avoid generating the vast majority of the 3^N eligible strings for longer lengths N . Significant savings arise from

the outset because, as we saw in Chapter 3, *either* the string $+-$ (and its negative $-+$) *or* the string $++$ (and its negative $--$) is always ruled out. A characteristic of our strategy is that it will not obtain in isolation the strings of precise length N , but must find all the strings from length 1 up to length N ; in practice this is the data we usually desire.

We describe the several aspects of this elimination step in the following four sections. In Section 4.1.1 we show how the geometric intersection test of Chapter 3 (which is our basic means of testing whether a string can appear in some admissible sequence) is converted from a mathematical result on the geometry of a configuration of parallelograms to a viable computer algorithm. Section 4.1.2 completes the implementation of the elimination process, and discusses further optimisations that can be achieved. In Section 4.1.3 we report on the runtime savings achieved by comparison with a naive algorithm that needs to consider all possible strings. We are also able to list some omissions from the corresponding results of Galias & Ogorzałek (1992) that our search has uncovered. Finally in Section 4.1.4 we outline a more sophisticated way of implementing the elimination step that entails considerable benefits by way of reducing storage requirements and improving runtime performance.

4.1.1 Implementation of the geometric intersection test

Since the strings are to be built by adding one digit at a time, at the point when we invoke the geometric intersection test to decide whether a string of length N can appear in some admissible sequence, we already have available the previously computed intersection for a length $(N - 1)$ substring. Our first task is to show how to use this existing information to compute *efficiently* the new intersection for the whole string of length N . Specifically, given a string $s_1 s_2 \dots s_{N-1}$ of length $(N - 1)$, together with its intersection S_{N-1} , where

$$S_{N-1} = w_0(\bar{I}^2) \cap w_1(\bar{I}^2) \cap \dots \cap w_{N-1}(\bar{I}^2) \quad (4.1)$$

and

$$w_r(\mathbf{x}) = \mathbf{A}'\mathbf{x} + \sum_{i=0}^{r-1} s_{N-1-i}\mathbf{A}'\mathbf{b}, \quad (4.2)$$

the problem is to find the intersection S_N corresponding to the string $s_0s_1 \dots s_{N-1}$, where the new digit s_0 has been added to the start of the string. The intersection S_N is

$$S_N = w_0(\bar{I}^2) \cap w_1(\bar{I}^2) \cap \dots \cap w_N(\bar{I}^2), \quad (4.3)$$

where the functions w_0, w_1, \dots, w_N are given again by the formula (4.2). Consequently from (4.1) we may write S_N as

$$S_N = S_{N-1} \cap w_N(\bar{I}^2), \quad (4.4)$$

and this allows us to determine S_N from S_{N-1} by computing the result of just a single intersection operation. (We remark that it is convenient here to denote by S_N the intersection set we represented as S' in Chapter 3.)

We showed in Chapter 3 that the set S_N arising from the intersection (4.3) is a convex polygon, so (4.4) represents S_N as the intersection of the convex polygon S_{N-1} and the parallelogram $w_N(\bar{I}^2)$. Our intention is to automate the geometric intersection test, and because the most difficult programming task in this is to compute the intersection of two sets, it is advantageous to transform (4.4) so as to arrange that any intersections to be computed arise in as simple a form as possible. For this purpose we write $S_N = w_N(T_N)$, thereby defining a new collection of sets T_0, T_1, \dots, T_N . Because the transformation w_N is an invertible affine map, the set S_N is empty if and only if T_N is, so the geometric intersection test may be equivalently stated in terms of the set T_N . From (4.4) we can express T_N in terms of T_{N-1} as

$$\begin{aligned} T_N &= w_N^{-1}(w_{N-1}(T_{N-1})) \cap \bar{I}^2 \\ &= A^{-1}(T_{N-1} - s_0 \mathbf{b}) \cap \bar{I}^2, \end{aligned} \quad (4.5)$$

and in this modified form the intersection to be computed at each stage is now with the fixed square \bar{I}^2 instead of the varying parallelogram $w_N(\bar{I}^2)$.

We remark at this point that we have based the computation of the geometric intersection on the use of the closed square \bar{I}^2 . In terms of the elimination of strings this relaxation

from I^2 to \bar{I}^2 is clearly a legitimate step, because to find the strings that generate admissible periodic sequences we subsequently search amongst those that are left. The gain is that we avoid the inconvenience of keeping track of the absent boundaries of I^2 throughout the intersection procedure.

As a result of formula (4.5), we can specialise to the task of computing the intersection between a convex polygon and the square \bar{I}^2 . To simplify the problem further, we view an intersection with \bar{I}^2 as the intersection with the four half planes $x_1 \leq 1$, $x_2 \geq -1$, $x_1 \geq -1$ and $x_2 \leq 1$. Since each half-plane is related to the previous one by a clockwise rotation of $\frac{\pi}{2}$ about the origin, it is sufficient to write computer code only for the intersection of a convex polygon with the half-plane $x_1 \leq 1$. The intersection with the square \bar{I}^2 is then found by calling this routine four times, making a rotation of the intermediate intersection between each call. Thus our code for the intersection with \bar{I}^2 is simply :

Intersection

Compute the intersection between a convex polygon P and the square \bar{I}^2 .

```
for (i=1; i<=4; i++) {
    ➡ Rotate  $P$  clockwise by  $\frac{\pi}{2}$  about origin.
    ➡ Find the intersection between  $P$  and the half-plane  $x_1 \leq 1$ .
}
```

Rotating a polygon by $\frac{\pi}{2}$ is straightforward : we store a convex polygon as a list of the coordinates of its vertices (taken in order around the polygon) so the rotation is achieved by transforming each vertex (x_1, x_2) to $(x_2, -x_1)$. The code for this is :

rotate

Rotate a polygon P by $\frac{\pi}{2}$ clockwise about the origin.

```
while (!empty(P)) {
    v = head(P); P = tail(P);
    rotated_P = add_to_list(vertex(v.x2, -v.x1), rotated_P);
}
return rotated_P;
```

This code has the side-effect of reversing the order in the list of the vertices of the polygon and we tolerate this for reasons of efficiency. When using list data-structures that are im-

plemented by a pointer from one element to the next, a much higher computational cost is incurred by appending an element to the end of the list than by inserting it at the start, and as a consequence efficient implementations of certain tasks naturally reverse the order of the elements. This does not cause any special difficulty for our lists, except that we must be aware that our representation of a polygon cannot then specify whether the vertices appear in clockwise or anticlockwise order around the polygon, so our code for computing the intersection with the half-plane must not rely on the order in which the vertices are presented.

We now describe an implementation for the fundamental task of finding the intersection Q of a convex polygon P with the half-plane $x_1 \leq 1$. Although much simpler than the original intersection (4.4), in common with most problems in computational geometry, it is not trivial to program. In particular we need to be aware of the following two requirements for any implementation :

- (i) The routine must operate correctly irrespective of whether the first (or last) vertex of P lies to the left or right of the line $x_1 = 1$, since P could be supplied with any one of its vertices as the first in the list.
- (ii) In forming the output Q it may be necessary to add extra vertices which were not present in P (a new vertex is introduced wherever an edge of P crosses the line $x_1 = 1$), and to exclude vertices of P (all the vertices of P lying to the right of $x_1 = 1$ are absent from Q).

The strategy we employ is best outlined by a specific example.

Example 4.1 : Computation of the intersection with the half-plane $x_1 \leq 1$

We will take for our example the seven-sided polygon P plotted in Figure 4.1(a) and represented by the following list of vertices :

$$P : (-5, 1), (-4, -3), (-1, -4), (3, -2), (5, 0), (2, 4), (-3, 4).$$

$\underset{a}{-5}$
 $\underset{b}{-4}$
 $\underset{c}{-1}$
 $\underset{d}{3}$
 $\underset{e}{5}$
 $\underset{f}{2}$
 $\underset{g}{-3}$

The intersection of P with the half-plane $x_1 \leq 1$ is the polygon Q , plotted in Figure 4.1(b).

Our method constructs the intersection Q vertex by vertex, considering in turn the edges ab, bc, cd, de, ef, fg and ga of P . The location of each edge relative to the line $x_1 = 1$ determines which vertices are added to Q to build it from an initially empty list. Table 4.2 details the action taken at each step, and shows how the method correctly finds Q , where

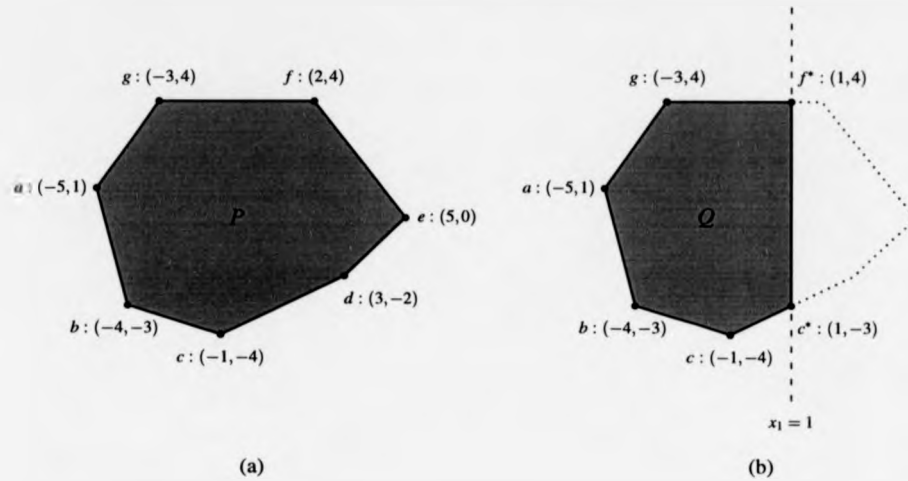
$$Q : \left(\underset{a}{-5, 1}, \underset{g}{-3, 4}, \underset{f^*}{1, 4}, \underset{c^*}{1, -3}, \underset{c}{-1, -4}, \underset{b}{-4, -3} \right).$$

The general strategy, producing the intersection of an arbitrary polygon P with the half-plane $x_1 \leq 1$, can be described by formalising the ideas arising in this example. Of course, it is vertices of P rather than edges which are read in (because of the way we have chosen to represent a polygon), so at a typical step of the computation where we consider the location of an edge $v_1 v_2$, the vertex v_1 has just been processed and its neighbour v_2 is the one currently being fetched. As our example illustrates, there are four different possibilities for the location of $v_1 v_2$ relative to the line $x_1 = 1$, namely

- (i) $v_1 v_2$ lies entirely to the left of $x_1 = 1$,
- (ii) $v_1 v_2$ crosses $x_1 = 1$ from left to right,
- (iii) $v_1 v_2$ lies entirely to the right of $x_1 = 1$,
- (iv) $v_1 v_2$ crosses $x_1 = 1$ from right to left,

and each requires a different action in terms of the construction of Q . Table 4.3 shows how these four possibilities are detected when given the pair of vertices v_1 and v_2 , and lists the action required in each instance to introduce the vertices of Q which lie on the edge $v_1 v_2$ of P . (Note that the vertex v_1 itself would not be added to the output at this stage, already having been processed as part of the previous edge.)

The decision process of Table 4.3 translates into the computer code shown in Listing 4.1. Adding a loop over the vertices of P gives a complete routine `left.restriction` to find the intersection of a polygon with the half-plane $x_1 \leq 1$, see Listing 4.2. Note that the recording of the first vertex is an essential step: the point is that it must be available again in order to process the final edge.

Figure 4.1: The polygon from Example 4.1, and its intersection with the half-plane $x_1 \leq 1$.

| Edge of P | Location of Edge Relative to $x_1 = 1$ | Corresponding Action | Output Q |
|-------------|---|---|---|
| ab | entirely to left | Add b to Q . | $(-4, -3)$ |
| bc | entirely to left | Add c to Q . | $(-1, -4), (-4, -3)$ |
| cd | crosses from left to right | Introduce new vertex c^* at crossing point. Add c^* to Q , but $d \notin Q$. | $(1, -3), (-1, -4), (-4, -3)$ |
| de | entirely to right | $e \notin Q$ so Q unchanged. | $(1, -3), (-1, -4), (-4, -3)$ |
| ef | entirely to right | $f \notin Q$ so Q unchanged. | $(1, -3), (-1, -4), (-4, -3)$ |
| fg | crosses from right to left | Introduce new vertex f^* at crossing point. Add f^* and g to Q . | $(-3, 4), (1, 4), (1, -3), (-1, -4), (-4, -3)$ |
| ga | entirely to left | Add a to Q . | $(-5, 1), (-3, 4), (1, 4), (1, -3), (-1, -4), (-4, -3)$ |

Table 4.2: The list of steps taken to find the intersection Q of the polygon P , from the figure above, with the half-plane $x_1 \leq 1$.

| Location of Vertices v_1 and v_2 Relative to $x_1 = 1$ | | Action Taken to Construct Q |
|---|-------|---|
| v_1 | v_2 | |
| left | left | Add v_2 to Q . |
| left | right | Add intercept of edge joining v_1 to v_2 with $x_1 = 1$ to Q . |
| right | left | Add intercept of edge joining v_1 to v_2 with $x_1 = 1$ to Q . Add v_2 to Q . |
| right | right | No action taken |

Table 4.3: The decisions and actions used to construct the vertices of Q arising from an edge $v_1 v_2$ of the original polygon P .

process_edge_v1v2

Construct vertices of Q arising from an edge $v_1 v_2$ of P .

```

if ( left (v2)) {
  if ( right(v1)) {
    Q = add_to_list(vertex(1.0, intercept(v1, v2)), Q);
    Q = add_to_list(v2, Q);
  }
  else
    Q = add_to_list(v2, Q);
}
else {
  if ( left(v1))
    Q = add_to_list(vertex(1.0, intercept(v1, v2)), Q);
}

```

Listing 4.1: The translation into computer code of the decision process from the table above. The routine calls three simple functions to test whether a vertex lies to the left, or right, of $x_1 = 1$, and to find the intercept of an edge $v_1 v_2$ with $x_1 = 1$.

left_restriction

Compute the intersection Q of a polygon P with the half plane $x_1 \leq 1$.

```
v2 = head(P); P = tail(P);
v_initial = v2;
```

```
while (!empty(P)) {
  v1 = v2; v2 = head(P); P = tail(P);
  ➡ Use process_edge.v1v2 to construct vertices of  $Q$  for the edge  $v_1 v_2$ .
}
```

```
v1 = v2; v2 = v_initial ;
➡ Use process_edge.v1v2 to construct vertices of  $Q$  for the final edge.
```

```
return Q;
```

Listing 4.2: The completed routine to compute the intersection of a polygon with the half-plane $x_1 \leq 1$.

4.1.2 Applying the test to find the strings of length N that appear within some admissible sequence

With the coding of the geometric intersection test accomplished, it is a relatively straightforward task to expand our implementation of the test into a complete program for finding all strings of length N which appear within some admissible sequence. We achieve this in the `next_length` procedure, shown in Listing 4.3. This procedure, when called with the complete list of strings of length $(N - 1)$ that appear in some admissible sequence (where, for each string, its associated polygon T_{N-1} is supplied also), computes all strings of length N that appear in some admissible sequence. Lists of strings of longer and longer lengths are then calculated simply by repeated calls to this procedure, feeding the output from one call back as the input to the next. The following paragraphs give a brief summary of the operation of `next_length`.

The input is a list of the strings of length $(N - 1)$ and their polygons T_{N-1} , presented as (string, polygon) pairs. For each string $s_1 s_2 \dots s_{N-1}$, three new ones of length N are created by adding the digits $-$, 0 and $+$ respectively at the start. The geometric intersection test is applied to the new strings (computing T_N from T_{N-1} as described in the previous section)

next.length

Given the list of strings of length $(N - 1)$ that appear within some admissible sequence, together with their associated polygons T_{N-1} , compute the corresponding list at length N .

```

digit = {'-', '0', '+'};

while (!empty(pairs_Nminus1)) {
  p = head(pairs_Nminus1); pairs_Nminus1 = tail(pairs_Nminus1);
  string_Nminus1 = p.string; T_Nminus1 = p.polygon;

  for (i=0; i<=2; i++) {
    string_N = augment(digit[i], string_Nminus1);

    ➡ Compute  $T_N = A^{-1}(T_{N-1} - s_0 \mathbf{b}) \cap I^2$  as described in Section 4.1.1.

    if (empty(T_N))
      delete_string(string_N);
    else
      pairs_N = add_to_list (pair(string_N, T_N), pairs_N);
  }

  delete_string(string_Nminus1);
  delete_polygon(T_Nminus1);
}

```

Listing 4.3: Code to compute the list of strings of length N that appear within some admissible sequence, from the list of those of length $N - 1$.

to determine if any of these may appear within some admissible sequence : any that qualify are stored in a new list of pairs to be returned as output, and the remainder are discarded.

The deletion statements appearing in the code are important because the storage of the strings and their associated polygons imposes a large burden on the computer memory. Indeed with this method it is ultimately the storage requirement rather than excessive running time that places an upper limit on the length of the strings that can be investigated. At most it is necessary to hold in memory two lists of strings, for once the list of strings of length N has been compiled, there is no further use for any of the strings of length $(N - 1)$ and they can be discarded. In fact the implementation of `next.length` requires less memory than this, since each string of length $(N - 1)$ is deleted as soon as the three new strings of

length N arising from it have been created. Thus in practice the amount of storage required does not exceed that needed to hold the list of strings of length N . Beyond emphasising the importance of reclaiming memory that is no longer needed, we will not pursue the subject of how the deletions are organised or achieved, because this is an issue linked more to the specific programming language employed than it is to the algorithm design itself.

To set the program running we must supply an initial list of strings and their associated polygons as the input to the first call of `next.length`. This initial input can be kept extremely simple, because by means of an appropriate interpretation for the set T_0 we can use `next.length` itself to calculate the strings of length 1. To deduce the appropriate initial input, we note that the set S_1 , used as the geometric intersection test for strings of length 1, is

$$S_1 = \mathbf{w}_0(\bar{I}^2) \cap \mathbf{w}_1(\bar{I}^2) = \bar{I}^2 \cap \mathbf{w}_1(\bar{I}^2) \quad (4.6)$$

where $\mathbf{w}_0(\mathbf{x}) = \mathbf{x}$ and $\mathbf{w}_1(\mathbf{x}) = \mathbf{A}\mathbf{x} + s_0\mathbf{b}$. Hence we can write the set $T_1 = \mathbf{w}_1^{-1}(S_1)$ as

$$T_1 = \mathbf{A}^{-1}(\bar{I}^2 - s_0\mathbf{b}) \cap \bar{I}^2, \quad (4.7)$$

and if we let $T_0 = \bar{I}^2$ this expression matches the general expression (4.5) for T_N in terms of T_{N-1} . Thus for our first call to `next.length` we may simply supply the pair consisting of the empty string and the square \bar{I}^2 .

We have now described the implementation of a complete program to find all strings of length N that appear as a substring within some admissible sequence. There is, however, a useful optimisation for the method which improves the speed of operation at a cost of only a modest amount of extra storage. The technique, mentioned in Galias & Ogorzałek (1992), involves keeping a record of all the *strings* that *cannot* appear in any admissible sequence (i.e. all those strings that fail the geometric intersection test). This list may then be used subsequently to speed up the elimination of many of the longer strings. For instance, when constructing the strings of length 2 from those of length 1 we may well learn that `++` cannot appear within any admissible sequence (indeed we saw in Chapter 3 that for any parameter

value $0 < \theta < \frac{\pi}{2}$, or equivalently $0 < a < 2$, this string is prohibited). This would be valuable information, because when building the length N strings from the length $(N - 1)$ strings, any contender starting with $++$ can immediately be discarded, without the need for the time-consuming geometric intersection test.

The checks needed to rule out strings by this means are easiest to describe in terms of a definite example, and by way of illustration we will consider a call to `next.length` to compute the strings of length 5 that appear in some admissible sequence, taking as input the corresponding list of strings of length 4. Assume that as part of the calculation from the previous calls to `next.length`, an exception table has been constructed containing all strings of lengths from 1 to 4 which *cannot* appear as a substring of any admissible sequence. Suppose that the string currently under scrutiny is $+0-0+$; then we check, against the table entries, the first four substrings at the start of $+0-0+$, i.e. the substrings $+$, $+0$, $+0-$ and $+0-0$. Should a match be found, we discard $+0-0+$ and commence again with a new string; only when none of the four substrings appears in the exception table do we need to resort to the geometric intersection test for $+0-0+$. Note that $+$, $+0$, $+0-$ and $+0-0$ are the only substrings which need to be checked because any other substring of $+0-0+$ is also a substring of $0-0+$, and $0-0+$ cannot contain any substring present in the exception table because we know that, as a string amongst the input for the current call of `next.length`, it appears in some admissible sequence.

Galias and Ogorzałek allude to this optimisation in their search, "to speed up the calculations", but we caution here against its overuse: without care an "optimised" program would run much more slowly than our basic method not employing any optimisation. The problem is that although checking each new string against *short* substrings does save effort (because a relatively large proportion of long strings can be ruled out in this way), less and less is gained by checking against longer substrings. Checks against long entries in the exception table are time-consuming and result in the elimination of few strings, so that the geometric intersection test has to be invoked in any case for the majority of strings being examined by `next.length`. So we must anticipate that it would be counter-productive to check substrings beyond a certain length. This is exactly what we found in practice. In

the graph of Figure 4.2 is plotted the time taken by a call of `next.length` to compute the strings of length 100 from those of length 99, against the maximum length of substring checked in the optimisation step. The graph clearly shows that (in terms of running time) it is detrimental to check other than the very shortest substrings. Figure 4.3 displays, magnified, that part of the graph close to the minimum point, and we see that fastest operation is achieved by checking substrings up to length 6. The reduction of the run-time is from 2.79 seconds (the run-time for the call to `next.length` without any optimisation) to 2.16 seconds, a saving of more than 20%. (By contrast, checking for substrings of all lengths slows the run-time down to 24.2 seconds, more than eight times slower than with no optimisation.) Our experiments have indicated that this optimal value, for the upper limit on the lengths of substrings checked against an exception table, shows virtually no variation either with the parameter value or with the length of strings being constructed. In view of all of this, in our program we employ a check for substrings up to a fixed length of 6.

4.1.3 Efficiency of the method and a comparison of the search results with Galias & Ogorzałek (1992)

Listing the strings which appear in admissible sequences is an intermediate step towards our main goal of finding the admissible periodic sequences, and as such it is not of great interest to examine the lists of strings themselves. However the *size* of the lists is a pertinent issue to our present discussion, since the rationale of our approach is that by generating these intermediate lists of strings we can bypass the majority of the 3^N strings of length N which could potentially generate admissible periodic sequences.

To give some measure of the size of the lists of strings generated by our method, we present in Table 4.4 the number of strings of length N which appear in some admissible sequence (as found by our `next.length` procedure) tabulated for $N = 10, 20, \dots, 200$. A glance at the table shows that a vast reduction in the scope of the original problem is achieved: the number of strings that now need to be scrutinised in the search for admissible periodic sequences is many orders of magnitude smaller than 3^N .

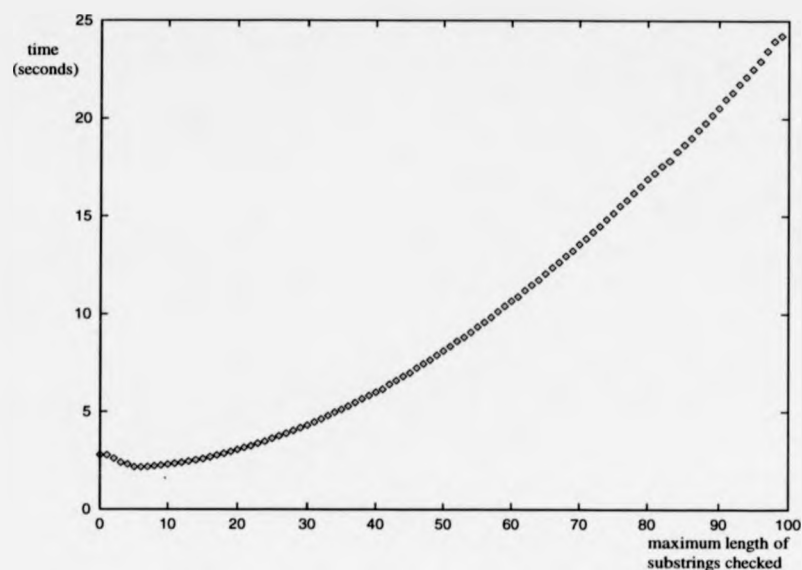


Figure 4.2: Variation in run time with the degree of "optimisation" employed. The plot shows the time to determine the strings of length 100, that appear in some admissible sequence, derived by starting from the list of those of length 99.

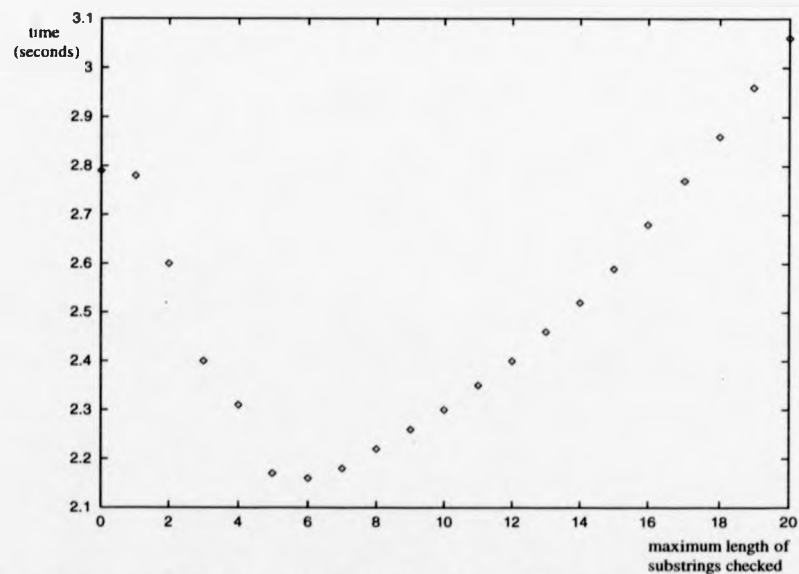


Figure 4.3: A magnification of the plot above. Fastest operation is achieved when the substrings of length up to 6 are checked against an exception table.

Of course any consequent reduction in computational effort must be set against the overhead incurred in producing our lists. Because of the inductive strategy employed in this phase of the overall task, the cost of this overhead for the length N strings is proportional to the size of the list of length $(N - 1)$ strings. The task of generating each length N string (given the list of length $(N - 1)$ strings that appear in admissible sequences) and applying the geometric intersection test to it within a call to `next.length` is essentially commensurate with the task of testing directly that a length N string generates an admissible periodic sequence (see Section 4.2). These two factors, together with the relatively small size of the lists, mean that there is a considerable gain in adopting our two-phase approach. The following example illustrates this point.

Example 4.2 : Computational effort required to process strings of length $N = 200$

We calculate the amount of work (measured in terms of the total number of strings processed) in finding all the strings of length 200 that generate admissible periodic sequences, after we have completed this task at length 199. From the last line of Table 4.4, we find that there are 103073 strings of length 200 that subsequently need to be investigated to see if they generate admissible periodic sequences. The overhead incurred in producing these from the list of strings of length 199 (which contains 101899 strings) is the examination of 3×101899 strings during the call to `next.length`. Processing the total $103073 + 3 \times 101899 = 408770$ to ascertain which strings of length $N = 200$ generate admissible periodic sequences compares very favourably with the method of checking directly all 3^N strings, since the latter entails processing this many strings as soon as N exceeds 11.

Our searches for the strings that appear in some admissible sequence revealed omissions in the results reported in Galias & Ogorzałek (1992). These authors presented a list of the strings up to length $N = 14$, at the parameter value $a = 0.5$, which *cannot* appear in any admissible sequence and which have the additional property that they contain no shorter substring that cannot appear in any admissible sequence. The point of such a list is to give a compact description of the strings which *do* appear in some admissible sequence; this latter collection of strings is very much larger, but can be generated from their list by a

| String Length (N) | Count of Strings Found | Approximate Value of 3^N |
|--------------------------|---------------------------|-------------------------------|
| 10 | 125 | 5.9×10^4 |
| 20 | 553 | 3.5×10^9 |
| 30 | 1309 | 2.1×10^{14} |
| 40 | 2503 | 1.2×10^{19} |
| 50 | 4231 | 7.2×10^{23} |
| 60 | 6455 | 4.2×10^{28} |
| 70 | 9139 | 2.5×10^{33} |
| 80 | 12535 | 1.5×10^{38} |
| 90 | 16627 | 8.7×10^{42} |
| 100 | 21187 | 5.2×10^{47} |
| 110 | 26393 | 3.0×10^{52} |
| 120 | 32153 | 1.8×10^{57} |
| 130 | 38497 | 1.1×10^{62} |
| 140 | 45391 | 6.3×10^{66} |
| 150 | 53053 | 3.7×10^{71} |
| 160 | 61581 | 2.2×10^{76} |
| 170 | 70831 | 1.3×10^{81} |
| 180 | 80763 | 7.6×10^{85} |
| 190 | 91519 | 4.5×10^{90} |
| 200 | 103073 | 2.7×10^{95} |

Table 4.4: The number of strings of length N which may appear as a substring within some admissible sequence, compared with the total number of strings of length N . The table has been compiled for the parameter value $a = 0.5$, but the numbers of strings found for other values across the parameter range $-2 < a < 2$ are comparable.

simple algorithm.

Table 4.5 is the corresponding output from our computer search. Our program will construct this type of list if we insist on checking every new string formed in next.length against a table of substrings up to length $(N - 1)$, i.e. the extreme case of the optimisation procedure described in the last section. The search results reveal additional strings at lengths $N = 10, 11, 12$ and 14 absent from Galias & Ogorzałek (1992): for instance, when $N = 10$, we find the six extra strings **0-00-0000+**, **+0000-00-0**, **000-00-000**, **000+00+000**, **-0000+00+0** and **0+00+0000-**.

As independent confirmation that these extra strings should properly be included in the

| <i>N</i> | List of Strings |
|----------|---|
| 1 | none |
| 2 | ++, -- |
| 3 | -0-, +0-, -0+, +0+ |
| 4 | -00+, 0+-0, 0-+0, +00- |
| 5 | 000+-, -000-, +000-, 0+-+0, -+000, +-000, 0-+-0, -000+, +000+, 000-+ |
| 6 | +-00-+, +0000+, -0000-, -+00+- |
| 7 | -00000-, +00000-, 0+-++0, 0-++-0, -00000+, +00000+ |
| 8 | -000000+, +000000+, 0+-++-0, 0-++-+0, -000000-, +000000- |
| 9 | -0000000-, +0000000-, -0000000+, +0000000+ |
| 10 | 0-00-0000+, -00000000+, +00000000+, 0+-++-+-0, +0000-00-0, 000-00-000, 000+00+000, -0000+00+0, 0-++-+-+0, -00000000-, +00000000-, 0+00+0000- |
| 11 | -00-00-+-+-, 000+00+00+-, -000000000-, -+-+00-00-, 0+-++-+-+0, -+00+00+000, 00000+00000, 00000-00000, +-00-00-000, 0-+-++-+-+0, +-+-+00+00+, +000000000+, 000-00-00-+, +00+00+-+-+ |
| 12 | +-00-00-00-+, -0000000000+, +0000000000+, -00-0000+00+, 0+-++-+-+0, 0-+-++-+-+0, +00+0000-00-, -0000000000-, +0000000000-, -+00+00+00+- |
| 13 | -00000000000-, +00000000000-, 0+-++-+-+0, 0-+-++-+-+0, -00000000000+, +00000000000+ |
| 14 | 000-00-00-00-+, -000000000000+, +000000000000+, 000-000000000+, 00000-0000+00+, 0+-++-+-+0, +000000000-000, +-00-00-00-000, +00+0000-00000, -00-0000+00000, -+00+00+00+000, -000000000+000, 0-+-++-+-+0, 00000+0000-00-, 000+000000000-, -000000000000-, +000000000000-, 000+00+00+00+- |

Table 4.5: The strings up to length $N = 14$, when $a = 0.5$, that cannot appear within an admissible sequence and which do not themselves contain a shorter substring which cannot appear within an admissible sequence. (Comparison of our list with a corresponding list given in Galias & Ogorzałek (1992) reveals that there are some omissions in the latter.)

list, we include a separate proof for the string **000+00+000** of length $N = 10$. A check against the strings of length 1 to 9 in Table 4.5 (or equivalently against the list in Galias & Ogorzałek (1992), which is identical over lengths 1 to 9) readily confirms that the string **000+00+000** contains no substring which has already appeared at a shorter length. Thus it remains to show that **000+00+000** cannot appear within any admissible sequence. By Proposition 3.2, the geometric intersection test, this is equivalent to showing that for this string the intersection

$$S = w_0(I^2) \cap w_1(I^2) \cap \dots \cap w_{10}(I^2)$$

is empty. We accomplish this by showing that there is no point which belongs to all three of the parallelograms $w_0(I^2)$, $w_4(I^2)$ and $w_{10}(I^2)$. When $a = \frac{1}{2}$, the coordinates of the vertices of these parallelograms are :

$$w_0(I^2) : \begin{matrix} (-1, -1), & (1, -1), & (1, 1), & (-1, 1) \\ A_0 & B_0 & C_0 & D_0 \end{matrix}$$

$$w_4(I^2) : \begin{matrix} \frac{1}{24}(-22, -47), & \frac{1}{24}(2, -19), & \frac{1}{24}(-26, -9), & \frac{1}{24}(-50, -37) \\ A_4 & B_4 & C_4 & D_4 \end{matrix}$$

$$w_{10}(I^2) : \begin{matrix} \frac{1}{2^{10}}(-718, -1755), & \frac{1}{2^{10}}(650, -2975), & \frac{1}{2^{10}}(1870, -997), & \frac{1}{2^{10}}(502, 223). \\ A_{10} & B_{10} & C_{10} & D_{10} \end{matrix}$$

and the three are plotted together in Figure 4.4. To complete the proof we note that $w_0(I^2)$ is contained in the half-plane $x_2 \geq -1$, and the set $w_4(I^2) \cap w_{10}(I^2)$ is contained in the intersection of the two half-planes whose edges are extensions of the line segments B_4C_4 and $A_{10}D_{10}$ respectively. The equations for these two half planes are $5x_1 + 14x_2 \leq -16$ and $989x_1 - 610x_2 \geq 352$, and the apex of the region of intersection, shown shaded in Figure 4.5, has coordinates $(-\frac{151}{528}, -\frac{1099}{1056})$, consequently it has no overlap with the half-plane $x_2 \geq -1$.

For reference we include as Table 7 in Volume II the continuation of Table 4.5 for string lengths up to $N = 40$, and there we demonstrate how the information in the table may be used to construct all the strings at each length N that appear in some admissible sequence. The fact that the number of strings in this table is very much smaller than the number of

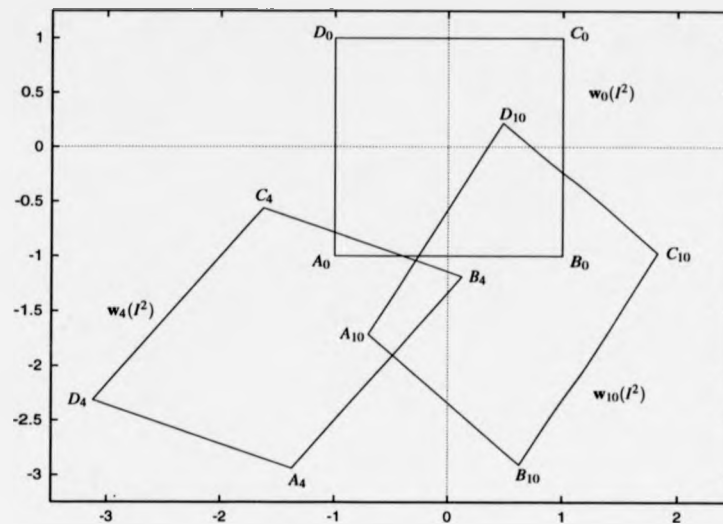


Figure 4.4: Three parallelograms $w_0(I^2)$, $w_4(I^2)$, and $w_{10}(I^2)$ arising from the string $000+00+000$. The parallelograms have no region of common intersection, verifying that this string cannot appear within any admissible sequence.

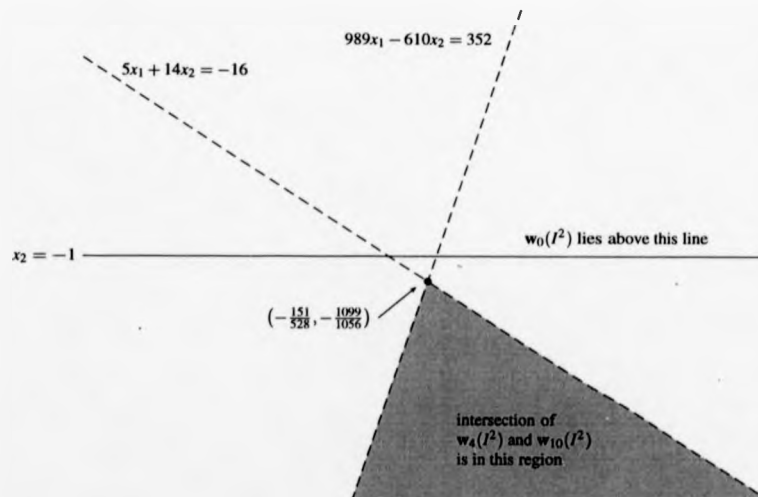


Figure 4.5: The three parallelograms have no region of common intersection, because $w_0(I^2)$ lies entirely above the line $x_2 = -1$ while the intersection of $w_4(I^2)$ and $w_{10}(I^2)$ lies entirely below.

those strings which appear in some admissible sequence (typically a few tens of strings in comparison with thousands) justifies our earlier remark that the optimisation method from Section 4.1.3 only makes modest demands on memory.

4.1.4 An alternative search strategy using tree-walking

The inductive strategy we adopted in Section 4.1.2 is not the only way to organise the computation and here we present an attractive alternative. The approach we now describe views the 3^N possible strings of length N (together with their associated polygons T_N) as nodes in the N^{th} layer of a ternary tree, and finds those strings which appear in some admissible sequence via a depth-first search over this tree.

We define the tree by letting the three children of each node be the strings formed by adding the digits $-$, 0 and $+$ respectively to the parent string. Thus for a node in the $(N-1)^{\text{st}}$ layer whose string is $s_1s_2 \dots s_{N-1}$, the three children are $-s_1s_2 \dots s_{N-1}$, $0s_1s_2 \dots s_{N-1}$ and $+s_1s_2 \dots s_{N-1}$. This relationship between parent and children allows us to re-use the code developed in Section 4.1.1 to compute the polygons T_N in inductive fashion as we travel downwards through the tree.

The tree structure fits naturally with the inheritance property of strings that appear within admissible sequences, and we use this, as before, to limit the amount of computation. Whenever we reach a node whose string $s_1s_2 \dots s_{N-1}$ fails the geometric intersection test, we may automatically rule out the entire subtree below this node (because every node of the subtree would contain the string $s_1s_2 \dots s_{N-1}$ as a substring). In this way large branches of the tree are pruned away, so the number of nodes to be visited during the depth-first search is vastly reduced from the potential maximum of 3^N per layer.

The depth-first search over the tree is ideally suited to a recursive implementation and is accomplished by the procedure `search_subtree` shown in Listing 4.4. This procedure takes as its argument a node of the tree consisting of a string $s_1s_2 \dots s_{N-1}$ together with its associated polygon T_{N-1} , and searches the entire subtree (of longer strings) below that node. The code can be extremely compact because each call merely has to construct and

search_subtree

Given the node of the tree associated with the string $s_1s_2 \dots s_{N-1}$ and polygon T_{N-1} , find all strings below this node which appear in some admissible sequence.

```

for (i=-1; i<=1; i++) {

    ➡ Augment string_Nminus1 with the digit -, 0 or + in turn to form string_N.
    ➡ Compute  $T_N = A^{-1}(T_{N-1} - s_0b) \cap I^2$  as described in Section 4.1.1.

    if (empty(T_N))
        delete_string (string_N);
    else {
        ➡ Output string_N
        if (N < max_length) search_subtree(string_N, T_N);
        delete_string (string_N);
        delete_vlist (T_N);
    }
}

```

Listing 4.4: Code to find recursively the strings that appear within some admissible sequence by a depth-first search over the associated tree.

test in turn the three children of the node; any that pass the geometric intersection test are then processed by a new recursive call. Note that it is necessary to specify a maximum depth for the search (i.e. the maximum length of strings to be considered), otherwise the process would continue indefinitely as the search probed deeper and deeper into the tree. A search over the whole tree as far down as this pre-specified level will find all the strings up to this length which appear in some admissible sequence, and the search may be started by a single call to `search_subtree` at the root node, i.e. supplying as argument the empty string together with the polygon $T_0 = I^2$.

We illustrate the operation of the depth-first search by showing how it identifies all strings of lengths $N \leq 3$ which appear in some admissible sequence, when the parameter value is $a = 0.5$. Figure 4.6 shows the extent of the tree to be searched; in the diagram those strings which fail the intersection test are shown in light type, and the subtrees below them are not drawn, being exempted from the search. The numbers appearing alongside each node refer to the order in which the nodes are visited as the depth-first search proceeds. Figure 4.7 exhibits the recursive call structure arising in the course of this search.

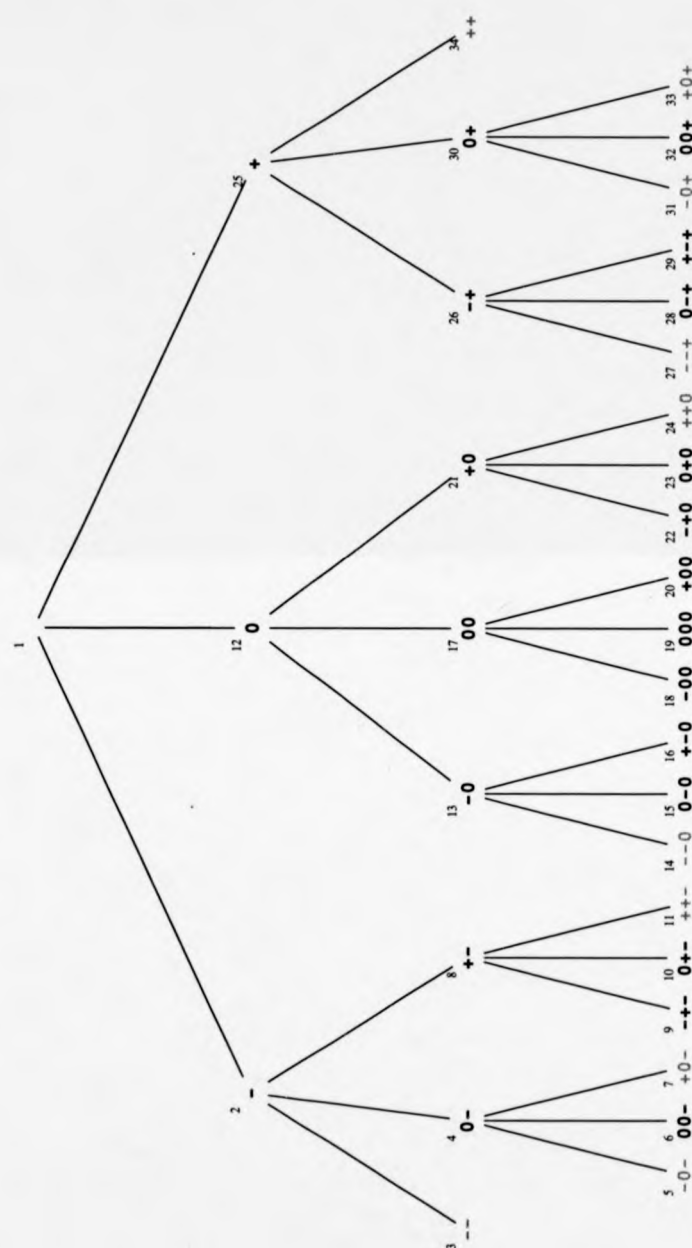


Figure 4.6: The tree corresponding to a depth-first search for strings of length up to 3, when $\alpha = 0.5$. The nodes are labeled with their corresponding strings; the associated polygons T_N are not shown.

```

search_subtree ()
  search_subtree (-)
    search_subtree (-- )
    search_subtree (0-)
      search_subtree (-0-)
      search_subtree (00-)
      search_subtree (+0-)
    search_subtree (+-)
      search_subtree (-+-)
      search_subtree (0+-)
      search_subtree (++-)
  search_subtree (0)
    search_subtree (-0)
    search_subtree (--0)
    search_subtree (0-0)
    search_subtree (+-0)
  search_subtree (00)
    search_subtree (-00)
    search_subtree (000)
    search_subtree (+00)
  search_subtree (+0)
    search_subtree (-+0)
    search_subtree (0+0)
    search_subtree (++0)
  search_subtree (+)
    search_subtree (-+)
    search_subtree (--+)
    search_subtree (0-+)
    search_subtree (+++)
  search_subtree (0+)
    search_subtree (-0+)
    search_subtree (00+)
    search_subtree (+0+)
  search_subtree (++)

```

no further recursive call : -- fails intersection test

no further recursive call : -0- fails intersection test

no further recursive call : maximum depth of recursion reached

no further recursive call : +0- fails intersection test

no further recursive call : maximum depth of recursion reached

no further recursive call : maximum depth of recursion reached

no further recursive call : +-+ fails intersection test

no further recursive call : --0 fails intersection test

no further recursive call : maximum depth of recursion reached

no further recursive call : maximum depth of recursion reached

no further recursive call : maximum depth of recursion reached

no further recursive call : maximum depth of recursion reached

no further recursive call : maximum depth of recursion reached

no further recursive call : maximum depth of recursion reached

no further recursive call : ++0 fails intersection test

no further recursive call : --+ fails intersection test

no further recursive call : maximum depth of recursion reached

no further recursive call : maximum depth of recursion reached

no further recursive call : -0+ fails intersection test

no further recursive call : maximum depth of recursion reached

no further recursive call : +0+ fails intersection test

no further recursive call : ++ fails intersection test

Figure 4.7: The recursive call structure for the depth-first search over the tree from Figure 4.6, corresponding to a left-to-right traversal.

The main attraction of using a depth-first search to find the strings is its economy with respect to memory usage. An implementation of the inductive search described in Section 4.1.2 requires a relatively large amount of free memory to run because the full list of strings of length $(N - 1)$ is presented as input to the `next_length` procedure in order to compute the full list of strings of length N . To give an estimate of the memory needed we note (from the results in Table 4.4) that when $N = 200$ the 103073 strings, each of 200 digits long, occupy $103073 \times 200 = 20$ Mbytes, not counting the extra storage for the polygon associated with each string. By contrast, the depth-first search implementation using the `search_subtree` procedure requires a bare minimum of memory (no more than would be needed to run any small computer program), and this is virtually independent of the length of the strings being searched for. The economy of memory usage with the depth-first search is achieved because the tree of strings is never physically built as a data-structure in memory : it exists only as the dynamically changing nested sequence of function calls. At any instant, the only strings actually present in memory are those which lie on the vertical path through the tree from the root node down to the node for the string currently under consideration. Thus the maximum number of strings stored at any instant is equal to the maximum depth of the tree, so that, for example, when finding the strings up to length 200 at most 200 strings are in memory simultaneously.

The depth-first search method affords a further opportunity to save on running time, in fact to cut it by a factor of two. We demonstrated in Proposition 3.11 that a string $s_0 s_1 \dots s_{N-1}$ passes the geometric intersection test if and only if its negation $\bar{s}_0 \bar{s}_1 \dots \bar{s}_{N-1}$ does. This means that there is no need to test any string in the right half of the tree, because each string found there is the negation of one from the left half. Since the depth-first search progresses from left to right over the tree (cf. the order in which the nodes are visited in Figure 4.6), we may interrupt the search at the half-way point (the last string which must be checked is $0 \dots 0$ with length equal to the maximum length for the search) and form the remaining strings cheaply by negating all those already found. With a little work the optimisation described in Section 4.1.1 can also be adapted for use with a depth-first search. Some modification is needed because the optimisation relied on the knowledge

of *all* short strings to reduce the demand for the geometric intersection test for longer strings; the difficulty with a depth-first search is that the strings are not found in order of increasing length. The problem can be circumvented by the use of a hybrid method : since the optimisation only needs a complete list of the *short* strings, we can make a preliminary search using the inductive strategy to find all the strings up to length 6 (and this is neither costly in time nor memory because the string length is so short), and then undertake the main task via a depth-first search, but with the list of short strings available from the outset for use in the optimisation.

There is one drawback of the depth-first search, and this is connected with the unusual order in which it generates the strings. The entire search must run to completion before we obtain the complete collection of strings at *any of the shorter lengths*. For example, in our search for strings of length $N \leq 3$ described above, the strings appear in the following order :

-, 0-, 00-, +-, --, 0+-, 0, -0, 0-0, +-0, 00, -00, 000, +00,
+0, -+0, 0+0, +, --, 0--, +-+, 0+, 00+

This makes it difficult to select an appropriate value for the maximum length of the strings in the search. (As we noted above, this must be set in advance for the recursion to have a limit, and a single value may not be appropriate over the whole of the parameter range.) On the one hand we want to set this maximum length as high as possible, to obtain information about long strings (and, ultimately, long-period admissible sequences), but equally a choice that results in an unfeasibly long computation is a disaster because if we are forced to terminate the computation before the depth-first search completes then we obtain no complete (and hence no useful) information regarding the strings of any length which appear in admissible sequences. In this respect, the inductive search method is superior, because the search for strings of length N is completed before any processing of the length $(N + 1)$ strings is undertaken; consequently we need only terminate a run of such a program when it exhausts computer resources, thereby obtaining information on sequences of as high a period as possible for the given parameter value.

4.2 The search for admissible periodic sequences at a fixed parameter value

4.2.1 Testing the eligible strings

We know, from the previous section, that to determine the strings of length N that generate admissible periodic sequences we can avoid searching amongst the full set of 3^N conceivable strings. Those that are produced there as eligible are sufficiently few in number to permit direct checking in order to determine which of them generate admissible periodic sequences. For computational testing it is effective to use the original criterion of admissibility given in Proposition 2.1; that is, the string $s_0 s_1 \dots s_{N-1}$ generates an admissible periodic sequence if and only if $\mathbf{x}_r \in I^2$ for $r = 0, 1, \dots, N-1$, where

$$\begin{aligned} \mathbf{x}_0 &= (\mathbf{I} - \mathbf{A}^N)^{-1} (s_0 \mathbf{A}^{N-1} + s_1 \mathbf{A}^{N-2} + \dots + s_{N-1} \mathbf{I}) \mathbf{b} \\ \mathbf{x}_1 &= (\mathbf{I} - \mathbf{A}^N)^{-1} (s_1 \mathbf{A}^{N-1} + s_2 \mathbf{A}^{N-2} + \dots + s_0 \mathbf{I}) \mathbf{b} \\ &\vdots \\ \mathbf{x}_{N-1} &= (\mathbf{I} - \mathbf{A}^N)^{-1} (s_{N-1} \mathbf{A}^{N-1} + s_0 \mathbf{A}^{N-2} + \dots + s_{N-2} \mathbf{I}) \mathbf{b}. \end{aligned} \quad (4.8)$$

To test a string $s_0 s_1 \dots s_{N-1}$ we commence by using

$$\mathbf{x}_0 = (\mathbf{I} - \mathbf{A}^N)^{-1} \mathbf{z}_0 \quad (4.9)$$

to calculate \mathbf{x}_0 , where

$$\mathbf{z}_0 = s_0 \mathbf{A}^{N-1} \mathbf{b} + s_1 \mathbf{A}^{N-2} \mathbf{b} + \dots + s_{N-1} \mathbf{b}, \quad (4.10)$$

and then we use the recurrence relations

$$\begin{aligned} \mathbf{z}_{r+1} &= \mathbf{A} (\mathbf{z}_r - s_r \mathbf{A}^{N-1} \mathbf{b}) + s_r \mathbf{b} \\ \mathbf{x}_{r+1} &= (\mathbf{I} - \mathbf{A}^N)^{-1} \mathbf{z}_{r+1} \end{aligned} \quad (4.11)$$

to determine the remaining points \mathbf{x}_r ($r = 1, 2, \dots, N-1$). The matrices \mathbf{A} and $(\mathbf{I} - \mathbf{A}^N)^{-1}$ and the vectors \mathbf{b} , $\mathbf{A}\mathbf{b}$, ..., $\mathbf{A}^{N-1}\mathbf{b}$ are independent of the particular string $s_0s_1\dots s_{N-1}$ being tested, so may be computed and stored before the search phase commences. This element of prior computation reduces the task of calculating \mathbf{x}_{r+1} from \mathbf{x}_r to two matrix-vector multiplications and two vector additions. Note also that it is frequently possible to terminate the computation prematurely because the string fails as soon as a point \mathbf{x}_r is found which lies outside of the square I^2 ; in these situations there is no need to compute any of \mathbf{x}_{r+1} , \mathbf{x}_{r+2} , ..., \mathbf{x}_{N-1} .

One might suppose it would be advantageous to use instead the Chua-Lin trigonometric inequalities, as derived in Section 2.1.4, and thus avoid the need for the matrix formalism. However this is not the case : such an approach incurs a higher computational cost because we lose the simple recurrence (4.11) that allows us to progress efficiently from one inequality to the next. With the scheme described above as the basis for the computer implementation we only need to calculate as a sum of N terms the single vector \mathbf{z}_0 , each of the subsequent \mathbf{z}_r 's and the \mathbf{x}_r 's can be produced in a fixed number of steps (i.e. independent of N). Thus the total number of arithmetic operations required to evaluate the expressions (4.8) for a given string $s_0s_1\dots s_{N-1}$ is proportional to the length N of the string. But with the Chua-Lin inequalities it is necessary to calculate the subject of each inequality as the inner product of a cycled version of the vector $(s_0, s_1, \dots, s_{N-1})^T$ with the constant vector $(\cos \frac{1}{2}N\theta, \cos \frac{1}{2}(N-1)\theta, \dots, \cos \frac{1}{2}(N-1)\theta)^T$, and the number of arithmetic operations required is proportional to N^2 .

4.2.2 A further reduction in the number of strings to be tested

When the eligible strings are produced by the inductive method of Section 4.1.2, the full list of strings of length N that may appear in some admissible sequence is available, and we can use this to reduce further the number of strings that need to be tested. A necessary condition for a string $s_0s_1\dots s_{N-1}$ to generate an admissible periodic sequence is that each of its $(N-1)$ cycled versions also appears as a substring within some admissible sequence.

This allows us to disqualify at once any string $s_0s_1 \dots s_{N-1}$, from the list of eligible strings, when one of its cycled versions is absent from the list.

Computer code to effect the elimination of those strings not having all their cycled versions in the eligible list is presented in Listing 4.5. It operates by removing strings in turn from the list and inspecting the cycled versions until one of the following three outcomes occurs :

- (a) *The i^{th} cycled version is not present in the list of candidate strings.* Thus the original string does not generate an admissible periodic sequence, no further action need be taken with it, and the $(i - 1)$ cycled versions already considered may be removed from the list of eligible strings.
- (b) *The i^{th} cycled version matches the original string.* In this case the original string is reducible so no further action is needed for it. The $(i - 1)$ cycled versions already considered may be removed from the list.
- (c) *All N cycled versions are present in the list of eligible strings.* The string must be forwarded for testing to determine whether it generates an admissible periodic sequence. All N cycled versions may be removed from the list.

After one of these outcomes, processing can continue with the next string from the head of the list. Since each new string processed causes *at least one* string to be removed from the list, and *potentially as many as N* strings, the whole task of elimination requires at most a single pass through the list of eligible strings.

By using this elimination procedure we have implicitly adopted the two conventions discussed in Section 2.4 on how we report the lists of strings that generate admissible periodic sequences, namely :

- (i) No reducible string $s_0s_1 \dots s_{N-1}$ is reported.
- (ii) Only one of the N cycled versions of a string $s_0s_1 \dots s_{N-1}$ which generates an admissible sequence of least period N is reported.

Both are desirable, since they prevent redundancy in the lists of output.

reduce_list

Given the list of eligible strings, eliminate those which do not have all N cycled versions also present in the list.

```

while (!empty(candidate_list)) {
    s = head(candidate_list); candidate_list = tail (candidate_list);

    i=1; cycled_s = cycle(s);
    while (i <= N-1
        && !( (i <= N/2) && (N % i == 0) && equal(cycled_s, s) )
        && present_in_list(cycled_s, candidate_list) ) {
        remove(cycled_s);
        cycled_s = cycle(s);
        i+=1;
    }

    if (i==N)
        ➡ Forward string s to test for admissibility.
}

```

Listing 4.5: Code to disqualify strings on the basis of the requirement that cycled versions must also be present.

When the elimination code described here is run, we find that it rules out virtually all strings from the lists of eligible candidates, to the extent that only a very few must be referred, frequently none at all, for direct testing for admissibility. A measure of the success of the elimination process is given in Table 4.6, which lists the number of eligible strings before and after disqualification on the basis of absence of cycled versions (when the parameter value $a = 0.5$). We remark, however, that this elimination process cannot entirely replace the test for admissibility. It is possible to have a string $s_0 s_1 \dots s_{N-1}$ for which all N cycled versions appear in some admissible sequence, but with the property that if a number of copies of the string are concatenated then there is a substring of this which cannot appear in any admissible sequence. An example is provided by the string **+00**, for we saw in Chapter 3 that

- **+00** and **00+** appear in some admissible sequence for all $0 < \theta < \pi$ (except at the isolated values $\frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \frac{2\pi}{3}$ and $\frac{3\pi}{4}$),

- $0+0$ appears in some admissible sequence for $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$,

but

- $+00+$ appears in some admissible sequence when $\frac{\pi}{3} < \theta < \frac{\pi}{2}$ or $\frac{3\pi}{4} < \theta < \pi$,

which means that for $\frac{\pi}{2} < \theta < \frac{2\pi}{3}$ the string $+00$ cannot generate an admissible periodic sequence (because the string $+00+$ is ruled out) even though all three cycled versions $+00$, $0+0$ and $00+$ are permitted.

Elimination through the absence of cycled versions is only a viable strategy provided we have a fast means of determining whether or not a given string is present in the list of eligible candidates. Making simple sequential searches through the list would be impossibly slow because an unsuccessful search (i.e. ascertaining that a given string is not in the list) would require accessing *every* item in the list, and a successful search is hardly any faster, on average half the items need to be accessed. Instead, to achieve the fast look-up needed, we employ a hash table to store the list of eligible strings. Our hash table consists of a large array to hold the strings together with a function that assigns each string to a location in the array in a pseudo-random fashion; fast look-up is achieved because, provided the array size is large by comparison with the number of strings, the majority of strings will be assigned, under the hash function, to distinct locations in the array, and to check if a string is present in the table it is necessary to consult only a single table location, or at worst a very few locations. Hash tables are a standard data-structure in the computer science literature and a thorough description of their implementation and use may be found in Sedgewick (1988) and Knuth (1973); we limit the present discussion to the following four points that relate to our specific application.

1. Because the number of eligible strings can be estimated in advance, we use an open addressing scheme (i.e. where each table entry holds a single string, rather than a linked list of strings), and a table large enough to accommodate comfortably all the strings. The size of the table that is to hold the strings of length N is set just larger than three times the number of eligible strings found at length $(N - 1)$. This guarantees

| Length | Eligible Strings | After Elimination | Admissible Sequences | Length | Eligible Strings | After Elimination | Admissible Sequences |
|--------|---------------------|----------------------|-------------------------|--------|---------------------|----------------------|-------------------------|
| 101 | 21675 | 2 | 0 | 151 | 53865 | 0 | 0 |
| 102 | 22175 | 4 | 0 | 152 | 54689 | 2 | 2 |
| 103 | 22681 | 0 | 0 | 153 | 55533 | 0 | 0 |
| 104 | 23191 | 0 | 0 | 154 | 56381 | 0 | 0 |
| 105 | 23709 | 2 | 0 | 155 | 57237 | 0 | 0 |
| 106 | 24233 | 2 | 0 | 156 | 58093 | 2 | 0 |
| 107 | 24769 | 0 | 0 | 157 | 58957 | 0 | 0 |
| 108 | 25305 | 0 | 0 | 158 | 59825 | 0 | 0 |
| 109 | 25845 | 0 | 0 | 159 | 60697 | 0 | 0 |
| 110 | 26393 | 0 | 0 | 160 | 61581 | 0 | 0 |
| 111 | 26947 | 0 | 0 | 161 | 62483 | 0 | 0 |
| 112 | 27507 | 0 | 0 | 162 | 63393 | 4 | 0 |
| 113 | 28071 | 0 | 0 | 163 | 64309 | 0 | 0 |
| 114 | 28641 | 4 | 2 | 164 | 65229 | 0 | 0 |
| 115 | 29215 | 0 | 0 | 165 | 66153 | 0 | 0 |
| 116 | 29789 | 0 | 0 | 166 | 67077 | 0 | 0 |
| 117 | 30379 | 0 | 0 | 167 | 68007 | 0 | 0 |
| 118 | 30969 | 0 | 0 | 168 | 68939 | 0 | 0 |
| 119 | 31559 | 0 | 0 | 169 | 69881 | 0 | 0 |
| 120 | 32153 | 0 | 0 | 170 | 70831 | 0 | 0 |
| 121 | 32755 | 0 | 0 | 171 | 71789 | 0 | 0 |
| 122 | 33363 | 0 | 0 | 172 | 72761 | 0 | 0 |
| 123 | 33979 | 0 | 0 | 173 | 73739 | 2 | 0 |
| 124 | 34611 | 0 | 0 | 174 | 74723 | 0 | 0 |
| 125 | 35247 | 0 | 0 | 175 | 75713 | 0 | 0 |
| 126 | 35889 | 0 | 0 | 176 | 76703 | 2 | 0 |
| 127 | 36531 | 0 | 0 | 177 | 77697 | 0 | 0 |
| 128 | 37177 | 0 | 0 | 178 | 78703 | 0 | 0 |
| 129 | 37831 | 2 | 0 | 179 | 79729 | 0 | 0 |
| 130 | 38497 | 0 | 0 | 180 | 80763 | 0 | 0 |
| 131 | 39171 | 0 | 0 | 181 | 81805 | 0 | 0 |
| 132 | 39847 | 2 | 0 | 182 | 82857 | 0 | 0 |
| 133 | 40523 | 0 | 0 | 183 | 83917 | 0 | 0 |
| 134 | 41199 | 1 | 1 | 184 | 84985 | 0 | 0 |
| 135 | 41877 | 0 | 0 | 185 | 86057 | 0 | 0 |
| 136 | 42567 | 0 | 0 | 186 | 87141 | 2 | 0 |
| 137 | 43269 | 0 | 0 | 187 | 88227 | 0 | 0 |
| 138 | 43973 | 2 | 0 | 188 | 89317 | 2 | 2 |
| 139 | 44677 | 0 | 0 | 189 | 90415 | 0 | 0 |
| 140 | 45391 | 0 | 0 | 190 | 91519 | 4 | 2 |
| 141 | 46119 | 0 | 0 | 191 | 92645 | 0 | 0 |
| 142 | 46853 | 0 | 0 | 192 | 93785 | 0 | 0 |
| 143 | 47609 | 2 | 0 | 193 | 94929 | 0 | 0 |
| 144 | 48371 | 4 | 0 | 194 | 96073 | 0 | 0 |
| 145 | 49133 | 0 | 0 | 195 | 97223 | 0 | 0 |
| 146 | 49899 | 0 | 0 | 196 | 98385 | 0 | 0 |
| 147 | 50673 | 0 | 0 | 197 | 99551 | 0 | 0 |
| 148 | 51459 | 0 | 0 | 198 | 100725 | 0 | 0 |
| 149 | 52249 | 0 | 0 | 199 | 101899 | 0 | 0 |
| 150 | 53053 | 1 | 1 | 200 | 103073 | 2 | 0 |

Table 4.6: The effectiveness of the elimination of eligible strings based on absence of cycled versions.

enough table locations, and in practice results in the table being a little over one third full when it holds all the strings. (The figures in Table 4.4 indicate that we should expect this.) Note that there is only a small memory penalty for making the hash table too large, since an empty table location is stored as a NULL pointer and occupies only a single word of memory. For a technical reason associated with the operation of the hash function used, the size of the hash table should also be a prime (cf. note 3).

2. Linear probing is used to resolve conflicts when two strings hash to the same value; this is the only strategy available to us, because of the large number of deletions to be performed. (See Appendix B to Volume II for a separate note on the deletion of entries from a hash table.)
3. We use a hash function h which maps a string with key k to $h(k) = k \bmod M$, where M is the size of the table. The key for a string is the binary number formed by concatenating the ASCII codes for each of its digits. (The binary values for the individual digits are $+$ = 00101011, 0 = 00110000 and $-$ = 00101101; they differ only in the final 5 digits so these alone are used in compiling the key.) The following code computes our hash function for a string whose digits are stored in the array $s[0]$ to $s[N-1]$:

```

h = s[0] % M;
for (i=1; i<N; i++)
    h = ((h * 32) + s[i]) % M;
return h;

```

4. The number of probes required on average for a successful and an unsuccessful search are given by Knuth :

$$\frac{1}{2} \left(1 + \frac{1}{1-\alpha} \right) \quad \text{for a successful search}$$

$$\frac{1}{2} \left(1 + \frac{1}{(1-\alpha)^2} \right) \quad \text{for an unsuccessful search}$$

where α is the load factor, i.e. the proportion of locations in the table which are

occupied. In our case, where the load factor of the table is approximately $\alpha = \frac{1}{3}$ at the outset (and this is the greatest loading because strings are removed as the method proceeds), these formulae predict the average number of probes as 1.25 and 1.625 respectively, and both figures are quite acceptable for our purposes.

Finally, we note that using a hash table to store the list of eligible strings does not mean we lose the notion of the head of the list which was used in the code (Listing 4.5), whereby we systematically select some next string for consideration, since we can access the table locations sequentially (via the array used to represent the table) as well as through the standard route using the hash function.

When the depth-first search strategy is used to isolate the eligible strings, the elimination techniques just described are not available, so we have no option but to test for the admissibility of each string individually. The difficulty arises because the order in which the strings are produced by the depth-first search means that the full list of eligible strings of length N is never available. (The apparent solution of storing all the eligible strings as they are generated, sorting the entire output by string length, and then passing the several lists of strings in turn to the elimination step, would entail unrealistic demands on disk storage; the storage needed for the whole collection of eligible strings up to length 200, at parameter value $a = 0.5$, is a little short of 1Gbyte.) Nevertheless, testing each string for admissibility as it is encountered by the depth-first search over the tree is a practical strategy provided that fast floating point arithmetic is used, and affords a means to find admissible periodic sequences with minimal memory usage. The speed of the arithmetic is the crucial factor in deciding which method to select. Indeed when we employ rational arithmetic to confirm the accuracy of our results (see Section 4.2.5) as an alternative to the more usual double precision floating point computation, the depth-first search method is unsuitable and it is appropriate to use the method of elimination on the basis of the absence of cycled versions, described here.

4.2.3 Calculating the intervals of admissibility

Each of the periodic sequences found by our search will typically be admissible not only for the parameter value at which the search was undertaken, but throughout an interval which contains the specific parameter value (see the discussion in Section 2.1.4). Once an admissible periodic sequence has been identified, it is a comparatively simple task to determine this interval.

For a string $s_0s_1 \dots s_{N-1}$ which has been shown to generate an admissible periodic sequence at a parameter value a , we know that the upper end-point of the interval of admissibility lies in the range $[a, 2]$. A bisection method may be used to refine this: given a range $[l, u]$ which contains this upper end-point, we test the string $s_0s_1 \dots s_{N-1}$ for admissibility at the midpoint value $m = \frac{1}{2}(u + l)$ and so decide in which of the half-intervals $[l, m]$ or $[m, u]$ the upper end-point is to be found. By repeating this bisection step, we can obtain an estimate for the location of the upper end-point to any desired accuracy. Code to implement this is

```

l = a; u = 2.0;
do {
  m = (u + l) / 2.0;
  if admissible (string, m)
    l = m;
  else
    u = m;
} while (u - l) > tolerance

```

which terminates with a range $[l, u]$, shorter in length than the specified tolerance, and containing the upper end-point. The lower end-point of the interval of admissibility can be located by a similar process, starting from an initial range $[-2, a]$. As a simple illustration, the string **+0000-0000** generates an admissible periodic sequence at $a = 0.5$ (cf. Table 4.7 in the next section), and the bisection method with the tolerance set to 10^{-10} gives the ranges for the upper and lower end-points:

lower end-point between $-2.91038304567 \times 10^{-11}$ and $4.36557456851 \times 10^{-11}$,
 upper end-point between 0.61803398421 and 0.618033984298,

resulting in an estimate for the interval of admissibility $[0, 0.618033984]$, which is correct to 9 decimal places.

It is important to point out that an estimate for an interval of admissibility obtained purely by the bisection method need not be correct if the sequence has two or more disjoint intervals of admissibility. An example causing the method to fail would be a sequence having two short intervals of admissibility $[l_1, u_1]$ and $[l_2, u_2]$ containing the parameter values a and $\frac{1}{2}(a+2)$ respectively. A bisection search for the upper end-point of $[l_1, u_1]$ would incorrectly converge to the upper end-point of $[l_2, u_2]$ instead, resulting in the much larger range $[l_1, u_2]$ being returned in place of $[l_1, u_1]$. The risk of a false result can be minimised by some additional testing : after the interval has been estimated by bisection, we select a large number of equally spaced (or randomly chosen) parameter values across the interval and verify that admissibility holds at each of these values, thus increasing our confidence that the sequence genuinely is admissible throughout the interval.

For our purposes, the possibility that an interval might rarely be incorrectly reported is not a serious concern since we use the results from the computer searches primarily as a guide to the identification of infinite families of admissible periodic sequences. As will be seen in Chapters 5 and 6, where we study a selection of these families, once the computer data has been used to pick out potential families we confirm the existence of these families and derive their associated intervals of admissibility by entirely different means, that is, a mathematical investigation of the Chua-Lin inequalities.

Using bisection to estimate the intervals of admissibility incurs a relatively high computational cost because the test for admissibility of a string must be conducted for a succession of different parameter values. Since these parameter values cannot be known in advance there is no alternative but to calculate the quantities $(I - A^N)^{-1}$ and $A\mathbf{b}$, $A^2\mathbf{b}$, ..., $A^{N-1}\mathbf{b}$ on demand. In practice, this is of little consequence because there are so few strings which generate admissible periodic sequences. Referring to Table 4.7 in the following section, we see that there are only 24 strings up to length 200 which generate admissible periodic sequences, at the parameter value $a = 0.5$, and this should be compared with the

list of 6306392 eligible strings. This applies more generally to any *post-processing* of the strings that generate admissible periodic sequences. Efficiency is crucial only in the primary search for them, where so many candidate strings have to be considered; thereafter using algorithms which are computationally more demanding presents no particular problem.

4.2.4 Results of the computer searches for admissible periodic sequences

The computer methods we have described exhibit excellent run-time performance and permit us to extend the scope of previously published searches : indeed we achieve the aim of reporting the admissible periodic sequences, not simply for an isolated parameter value as previously, but for the whole parameter range. The tables in the second volume of this thesis present our lists of strings which generate admissible periodic sequences, tabulated up to length 200 at increments of 0.2 for the parameter θ , over its entire range 0 to π . In fact our searches were conducted for a much finer subdivision of the θ range, but due to limitations of space we have been unable to tabulate the majority of our results in Volume II. Our search results are reported in terms of θ rather than a , where $a = 2\cos\theta$, since this is the more natural parametrisation when we present our theoretical results over the succeeding chapters. The lists provide an atlas of the admissible periodic sequences. Reading through, there were revealed some notable families of sequences at specific places within the parameter range, the starting point for topics investigated later in this thesis.

Every string listed in the tables appears together with the associated interval for θ over which it generates an admissible periodic sequence, this interval being estimated by the bisection method outlined in the previous section. Frequently an estimate for an interval end-point is found to lie very close to a simple rational multiple of π . The detection of this possibility is easily incorporated into our interval finding procedure by checking whether an estimate lies close (i.e. within some specified tolerance) to a low rational multiple of π , and those that do have been indicated accordingly in the tables. Later, when we undertake

a theoretical study of the Chua-Lin inequalities and isolate families of admissible periodic sequences, we are able to confirm in many instances that the end-points of their intervals of admissibility do coincide with the rational multiples of π signalled by our computer investigation. We present some results of this nature in Chapters 5 and 9.

For the purpose of comparison with the computer searches reported in the published literature, Table 4.7 lists all the strings of length $N \leq 200$ that generate admissible periodic sequences when the parameter value is $a = 0.5$. All the previously reported studies (Chua & Lin, 1988, 1990b; Wu & Chua, 1993; Galias & Ogorzałek, 1992) have been undertaken for this single choice of parameter value. Our list confirms the findings of these earlier studies, and our computational methods have allowed us to extend considerably the scope of the search; for the parameter value $a = 0.5$ we present the strings up to length $N = 200$, the maximum length previously reported was $N = 56$.

Having gathered data for the admissible periodic sequences across the whole parameter range, it has become clear that seeking the admissible periodic sequences associated with the parameter value $a = 0.5$ (aside from having a role as a "benchmark" case to test and compare various computer search strategies) is not particularly illuminating in terms of the results of the search. By comparison with many of the other parameter values we selected, we found that short-period admissible sequences are relatively scarce when $a = 0.5$, and moreover, that there are no obvious patterns discernible amongst the sequences uncovered. The second collection of strings we present here, shown in Table 4.8, are the results of the computer search at a parameter value just above $\theta = \frac{\pi}{3}$. Here are many more short-period sequences, all of broadly similar form, and we can assign each to one of a small number of collections according to the particular pattern of $+-$ pairs and zeros, suggesting the existence of larger families of admissible periodic sequences. Specifically we identify two such collections (admitting also their digit-by-digit negations),

```

N = 9   :  +--+00000
N = 21  :  +--++-+--+00000000000
N = 33  :  +--++-+--++-+--+00000000000000000
etc.
```

```

0 : 0
1 : +-
2 : +-
3 : +-
4 : +-
5 : +-
6 : +-
7 : +-
8 : +-
9 : +-
10 : 000-0000+0
11 : 00-+-+000+00+-+00-
12 : 00-+-+000+00+-+00-
13 : 00-+-+000+00+-+00-
14 : 00-+-+000+00+-+00-
15 : 00-+-+000+00+-+00-
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194 : 00-+-+000+00+-+00-
195 : 00-+-+000+00+-+00-
196 : 00-+-+000+00+-+00-
197 : 00-+-+000+00+-+00-
198 : 00-+-+000+00+-+00-
199 : 00-+-+000+00+-+00-
200 : 00-+-+000+00+-+00-

```

Table 4.7: The strings of length $N \leq 200$ which generate admissible periodic sequences when $a = 0.5$.

and

```

N = 15 : +-+--+00000000
N = 27 : +-+--+00000000000000
N = 39 : +-+--+000000000000000000
etc.

```

having the patterns $(+-)^{3m-1}0^{6m-1}$ and $(+-)^{3m}0^{6m+2}$, for $m = 1, 2, \dots, 16$. These will be the focus of further investigation in Chapter 5. Aside from these collections there remain in the table only seven additional strings, at lengths $N = 84, 108, 144, 156, 168, 180, 192$, and these likewise exhibit a common structure, formed from a run of $+-$ pairs and a run of $-+$ pairs separated by two runs of zeros.

As well as the families near $\frac{\pi}{3}$, we found large collections of sequences for parameter values θ near $0, \frac{\pi}{2}$ and π . These locations appear as broad peaks in the graph of Figure 4.8, where the number of strings, of length not exceeding 200, found to generate an admissible

Table 4.8: The periodic sequences admissible for a parameter value θ just greater than $\frac{\pi}{3}$, listed together with their intervals of admissibility.

Table 4.8: continued.

```

171 : #####
(1.04719755, 1.04786164) = (q, --)
#####
(1.04719755, 1.04786164) = (q, --)
#####

177 : #####
(1.04719755, 1.04781877) = (q, --)
#####
(1.04719755, 1.04781877) = (q, --)
#####

180 : #####
(1.04719787, 1.05173953)
#####
(1.04719806, 1.05033197)
#####

183 : #####
(1.04719755, 1.04777992) = (q, --)
#####
(1.04719755, 1.04777992) = (q, --)
#####

189 : #####
(1.04719755, 1.0477446) = (q, --)
#####
(1.04719755, 1.0477446) = (q, --)
#####

192 : #####
(1.04719787, 1.05121763)
#####

195 : #####
(1.04719755, 1.0477124) = (q, --)
#####
(1.04719755, 1.0477124) = (q, --)
#####

```

Table 4.8: continued.

periodic sequence is plotted against the parameter value θ . One might be concerned that these peaks are an artifact of the arbitrary limit that one must impose on the maximum length of the strings considered. To oppose this, we suggest that the dependence of relative frequency upon θ is essentially the same whether we base our plot on the counts of sequences having string length up to 50, 100, 150 or 200, for this see Figure 4.9. Horizontal resolution in the plot is limited by the spacing of the progression of parameter values at which we performed a computer search; the points are plotted at increments of 0.01 for θ . The number of sequences found at values of θ near zero and near π are particularly large, to the extent that the graph far exceeds the top of the plotting range at its left and right ends. Much can be said about the periodic sequences which are admissible for these values of θ , and this is the subject of Chapters 7, 8 and 9. Those sequences responsible for the peak around $\frac{\pi}{2}$ are somewhat simpler to study, and they form the first topic for our analytical

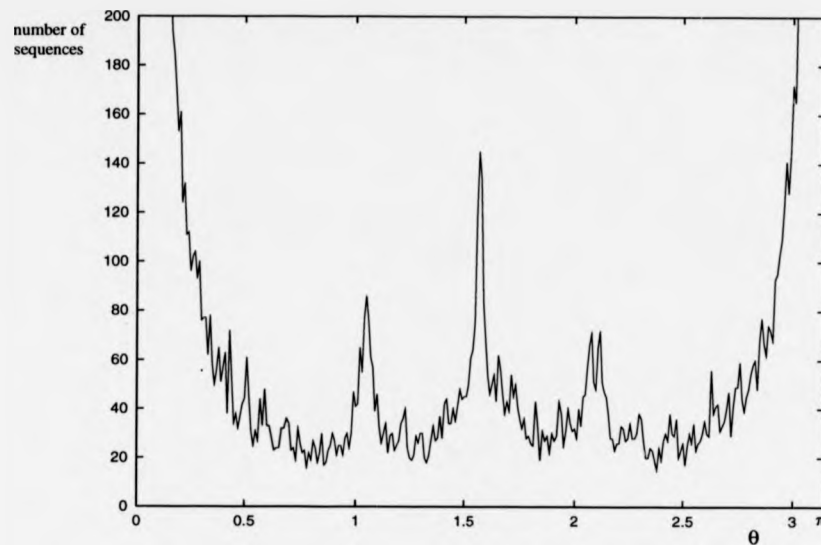


Figure 4.8: The number of admissible periodic sequences with period ≤ 200 plotted against the parameter value θ .

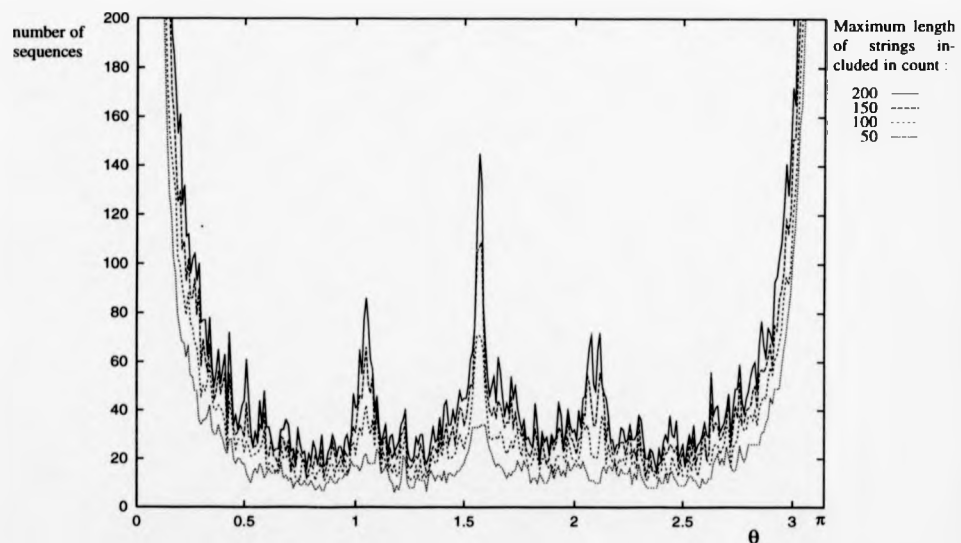


Figure 4.9: The graph from Figure 4.8, but plotted for string lengths up to 50, 100, 150 and 200; all four curves exhibit the same trends.

investigations, presented in Chapter 5. (Table 2 in Volume II contains the data from which Figure 4.8 was generated.)

Although our searches across the whole range of parameter values discloses a fuller picture of the great variety of admissible sequences which exist for the nonlinear map F , it is important to be aware that this type of investigation has its limitations. The first difficulty is to know how to select the values of θ at which the searches are conducted. In the absence of any prior information we chose a uniform distribution across the θ range $(0, \pi)$, but one could certainly select alternative sampling schemes, perhaps with the aim of conforming more to the distribution of numbers of admissible periodic sequences as θ varies. A more serious issue is that however small we take our sampling interval for the parameter value, there remains the possibility that whole families of sequences could still escape notice. Two factors that suggest this are :

1. The curve in Figure 4.8 shows significant local variation for all the degrees of horizontal resolution plotted. (We pursued this to $\delta\theta = 0.0001$.) It is possible that the curve is fractal in nature, and thus has local variation at all scales.
2. Several of the strings occurring in the tables in Volume II have extremely short intervals of admissibility, suggesting that they have been fortuitously chanced upon at that particular parameter value, and that equally many more will have been missed.

Indeed we do have evidence of a complete family that is missed by our search. Chapter 6 is devoted to a study of the strings having the specific pattern $+00+0\dots 0$ (two $+$ digits separated by two runs of zeros, one of length 2 and the other of even length greater than 2). There we show that there is an *infinite* family of strings of this type that generate admissible periodic sequences. Nevertheless all of its members are missed, none appears in any of the lists of Table 1 in Volume II, and none was encountered in our more extensive searches at parameter values spaced by 0.01. The sequences do exist nonetheless, Table 4.9 gives the results of a search when the parameter value is $\theta = 1.1828$ and includes the first string from this family, the string $+00+000000000000000000$, of length $N = 22$, associated with the tiny interval of admissibility $(1.18266629, 1.1830862)$. Thus we have demonstrated

that such strings can be picked up by our computer search, but only when we know exactly where to look. The pitfalls inherent in searching at fixed parameter values motivate the alternative we propose in Section 4.3. There we introduce a computational strategy to uncover *all* short length strings that generate periodic sequences that are admissible across *any part* of the parameter range $(0, \pi)$, a very much more challenging task.

4.2.5 Issues of numerical accuracy and the use of an exact arithmetic system

Up to this point in our presentation we have not mentioned the issue of precisely how the arithmetic calculations in our computer search strategies are performed. In fact, the tables of results reported in the last section, and also those in Volume II (apart from Table 6), were all obtained by implementations based on standard double precision arithmetic. Our programs use IEEE 754 double precision quantities and arithmetic operations throughout, the only exception being one higher-level mathematical function, a single call at the outset of the search to the cosine function to convert the supplied value of the parameter θ to the corresponding value for the parameter a , and for this we rely on the C library function.

By its nature, floating-point arithmetic cannot be completely accurate, and in view of this we now address the question of whether arithmetic inaccuracies cast doubt on the reliability of our search results. One might be concerned, particularly where the longer strings are involved, about the possibility of numerical inaccuracies provoking an incorrect decision as to whether a point lies to the left or right of the line $x_1 = 1$ whilst calculating the geometric intersections. Such an error could conceivably lead to an admissible periodic sequence being missed, or alternatively a spurious one reported. To demonstrate that there is potential for loss of numerical accuracy during the calculation, we draw attention, by way of example, to the size of a^{200} , a quantity linked with the calculation of the matrix power A^{200} . When $a = 0.5$ the number a^{200} is of the order 10^{-61} , and if this, say, was added to the value 1.0, it would be lost when only 15 or so digits are available in double precision arithmetic. By itself the disappearance of small quantities at this scale need not necessarily

```

1 : 0
2 : (5.31660401e - 07, 3.14159209) = (0, π)
3 : 0
4 : (5.31660401e - 07, 1.57079633) = (0, π/2)
5 : 0
6 : (1.04719755, 1.57079633) = (π/2, π/2)
7 : 0
8 : (1.04719755, 1.57079633) = (π/2, π/2)
9 : 0
10 : (1.15822212, 1.3962634) = (-π, π/2)
11 : 0
12 : (1.18266629, 1.1830862)
13 : 0
14 : (1.18266629, 1.1830862)
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233 : 0
234 : (1.18266629, 1.1830862)
235 : 0
236 : (1.18266629, 1.1830862)
237 : 0
238 : (1.1
```

Table 4.9: Results of the computer search at $\theta = 1.1828$, and which include the periodic sequence generated by $+00+000000000000000000$.

lead to arithmetic problems; real difficulties arise largely because of the subtraction of nearly equal quantities, when catastrophic cancellation can occur. Unfortunately any kind of numerical error analysis would appear to be a challenging undertaking, because the search algorithms are somewhat complicated, and the input data varied. For this reason, we adopt an alternative approach, and endeavour to validate our floating-point results by re-writing the computation entirely in terms of rational quantities that have an exact machine representation.

Using an implementation of our computer search methods based on rational arithmetic, we have re-calculated the admissible periodic sequences of length up to 200 for the parameter value $a = 0.5$. The list of strings which generate these sequences we found to be identical with that presented in Table 4.7. A more stringent validation of our earlier results comes from a check of the output from the first phase, that is the number of strings of length N that appear in some admissible sequence. Over the range tested, $1 \leq N \leq 200$, the numbers match exactly with those results we obtained using double precision calculations. For reference, the complete list of these confirmed values is included in Table 6 of Volume II.

Of course the drawback of using rational arithmetic is the attendant run-time costs, both for the speed of execution, where we lose the advantage that floating point operations are supported directly by the hardware, and also for the increased memory requirement to store large numerators and denominators of the rational quantities. The demanding task of finding the admissible periodic sequences makes it essential to employ a fast rational arithmetic implementation: for our searches we used the GNU Multiple Precision Arithmetic Library (Granlund, 1996). The time taken to compute the list of strings generating admissible periodic sequences at $a = 0.5$ was increased by a factor of 25, from 7 minutes to around 3 hours, but this figure is still within feasible limits to permit more extensive searching using rational arithmetic. Contrary to the situation for searches conducted using double precision arithmetic, the inductive search method is the best choice for speed of execution when rational quantities are involved, since the methods of Section 4.2.2 virtually eliminate the need to test for admissibility directly, which shifts the balance of computation away from slow arithmetic operations, replacing them with extra table look-ups and string processing.

We make two final comments regarding rational arithmetic. A difficulty is that moving from floating point to rational arithmetic causes some inconvenience in that we cannot easily verify our search results where we have specified the parameter $a = 2\cos\theta$, of the nonlinear map, through a particular θ value, since we cannot correspondingly know the rational value to assign to a . But a positive aspect is that in the context of rational arithmetic it is possible to determine, when applying the test for admissibility described in Section 4.2.1, whether the points x_0, x_1, \dots, x_{N-1} lie on the boundary of I^2 , so that the more delicate issue of admissible periodic sequences attributable to boundary points of I^2 could be investigated. This can never come within the scope of computer investigations based on floating point arithmetic.

4.2.6 Remarks on the execution times and computational complexity of the search methods

The search methods described in this chapter employ techniques from mainstream computer science known to be effective in accelerating the operation of computationally demanding tasks, namely the storage of intermediate results (also known as dynamic programming) used in the inductive search, and tree-pruning used in the depth-first search. Although techniques of this nature do not alter the underlying exponential dependence of the running time on the string length N , by effecting a reduction in the rate of that exponential growth, they can make a significant practical difference in terms of the size of the problem which can be solved. Using exhaustive string checking, the greatest string length explored was 25 (Wu & Chua, 1993), and essentially this limit is too low to reveal any patterns amongst the admissible periodic sequences identified. By contrast, we have been able to increase the maximum length of the strings investigated to 200, and our searches execute sufficiently fast, using only modest computer resources, to allow repeated runs to this length over the whole parameter range.

To obtain a measure of the improvement achieved by our search strategies by comparison with a simple exhaustive string checking algorithm, where run times increase with

string length N as 3^N , we have recorded the execution times to find the admissible periodic sequences at the parameter value $a = 0.5$. Figure 4.10 is a plot of this data, showing the time taken to find all admissible periodic sequences generated by strings from length 1 up to length N , for each value of N between 1 and 200. These timings were obtained for a double-precision version of our hybrid depth-first search method running on a Sun UltraSparc workstation. Performing a least-squares fit of the exponential curve $\alpha(e^{\beta N} - 1)$ to these times we obtained the parameter estimates $\alpha = 5.9200$ and $\beta = 0.021648$. (The goodness-of-fit statistic is $R^2 = 0.99594$, confirming that the curve is a reliable fit to the data.) This estimate for β corresponds to a time dependency on N of order 1.022^N , clearly a vast improvement over the 3^N cost of an exhaustive checking algorithm. In Table 4.10 we report the execution times for the searches, using the above formula to predict the time which would be needed to search to longer string lengths, and thus providing some estimate for the maximum string length which could be undertaken by our searches.

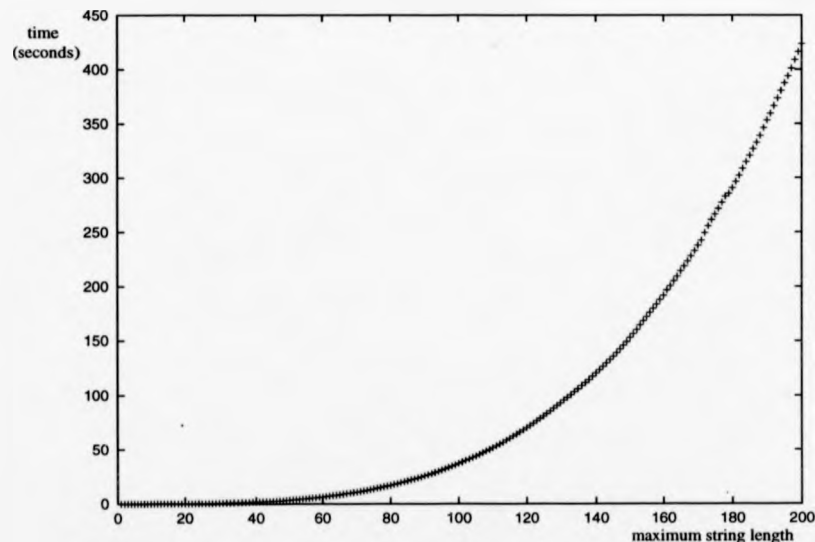


Figure 4.10: The execution times for the computer search for admissible periodic sequences. The graph shows the time taken plotted against the maximum string length.

| N | Execution Time | |
|-----|----------------|-------------|
| 50 | 3.40 seconds | (measured) |
| 100 | 37.15 seconds | (measured) |
| 150 | 153.03 seconds | (measured) |
| 200 | 423.67 seconds | (measured) |
| 250 | 22.01 minutes | (predicted) |
| 300 | 65.16 minutes | (predicted) |
| 350 | 3.21 hours | (predicted) |
| 400 | 9.48 hours | (predicted) |

Table 4.10: Computer time required to search for all admissible periodic sequences up to the specified maximum string length.

While techniques such as tree-pruning are effective in permitting us to investigate the admissible periodic sequences up to string lengths of a few hundred digits, no amount of continued adjustment to this type of algorithm can compensate for an exponential dependence of execution time on the string length. Thus the strategies proposed in this chapter could never enable us to deal with really large problem sizes where, say, the string lengths were in the order of tens of thousands of digits. Attempting a search at these lengths would only be feasible if one had a truly fast algorithm for finding the admissible periodic sequences, and in practice this would mean one whose execution time has a polynomial dependency on the string length.

As for the more fundamental question of whether a fast algorithm could exist, Chua & Lin (1990b) state that the problem of finding admissible periodic sequences is NP, that is, if we had a (notional) computer capable of non-deterministic operations then the search could be performed in polynomial time. The validity of this statement depends crucially on how the task is formulated. For example, on the available evidence, it appears that the problem of finding the strings which appear within some admissible sequence is inherently a problem incurring an exponential time cost, irrespective of whether non-deterministic computation is available to us. This is because the number of strings of length N appears to grow in an exponential fashion with N , so simply listing the strings will mean that the algorithm has an exponential time dependence. Problems which do belong to the class

NP are typically decision-based in nature and, for instance, one formulation of our search problem which would genuinely belong to NP is

“Is there a periodic sequence of least period N admissible at the (given) parameter value θ ?”.

An algorithm to solve it in polynomial time would be to use non-determinism to select a string of length N and subsequently check that it does generate an admissible periodic sequence via the Chua-Lin inequalities. Of more practical interest is whether this latter problem can be classed as one of the so-called NP-complete problems, a collection of well-known tasks of equivalent computational difficulty, for which no efficient methods of solution are known. Certainly the task of finding a string which satisfies the Chua-Lin inequalities has similarities with the problem of finding an assignment of variables to satisfy a boolean expression, itself one of the standard NP-complete problems.

4.3 A comprehensive search for admissible periodic sequences with short generating strings

The computer searches presented so far in this chapter provide us with very many examples of admissible periodic sequences for the map F , but, as we noted at the end of Section 4.2.4, searches performed for a uniformly distributed selection of values of the parameter θ across its range $(0, \pi)$ are not guaranteed to uncover all of the short strings that generate admissible periodic sequences. The difficulty is that a sequence can be missed entirely if its interval of admissibility lies wholly within the gap between two adjacent values of θ at which we perform computer searches. Regardless of how narrow a spacing we take between neighbouring values of θ , we cannot ensure that no admissible periodic sequences will be missed. In this section we outline an alternative computational method : although restricting us to much shorter string lengths than were achieved when searching at a fixed parameter value, it does locate *all* the strings that generate admissible periodic sequences of a given length and reports them together with their ranges of admissibility.

To undertake a comprehensive search of this nature we must address two new and fundamental problems. These arise because the geometric intersection test and the test for admissibility, which formerly were the basis for eliminating the majority of strings from the search and the means of deciding admissibility for the remainder, apply only at a specified value for the parameter θ and consequently cannot be used when we have no knowledge of the values of θ , if any, at which a string generates an admissible periodic sequence. Thus before we may attempt an implementation of a comprehensive search we must address the following :

- (i) How can we test whether a string $s_0s_1 \dots s_{N-1}$ generates an admissible periodic sequence over some (as yet unknown) range of values of θ ?
- (ii) What means do we have to limit the number of strings in order to avoid the need to test (in the manner of (i)) each of the 3^N conceivable strings of length N ?

Only the second of these questions is addressed in this section; it is convenient to defer a discussion and resolution of (i) until Chapter 6 which provides a better context for the issues that arise. Assume now that we have a procedure to decide whether there are parameter values where a string $s_0s_1 \dots s_{N-1}$ generates an admissible periodic sequence and to find the range of those values. Although this procedure will play the role that the test for admissibility assumed earlier, we need to be mindful that, unlike the simple computations needed for the admissibility test, it is costly and should be employed only when a string cannot be eliminated by alternative means.

Knowing the outcome of the searches at specified values of θ across the range $(0, \pi)$ (reported in Section 4.2.4), we may anticipate that the vast majority of the 3^N possible strings of length N will not generate an admissible periodic sequence at *any* value of θ in the range $(0, \pi)$. But we cannot disqualify strings in the same manner as previously because there seems little or no possibility of adapting the geometric intersection test to determine whether the intersection remains empty for all parameter values θ from 0 to π . Fortunately the very useful inheritance property is still available to us, in the form that if a string $s_0s_1 \dots s_{N-1}$ cannot appear in any admissible sequence over some range of parameter

values, then any string containing $s_0s_1 \dots s_{N-1}$ as a substring is likewise disqualified over that range.

The approach we take is to consider short length strings whose ranges for θ where they can appear in some admissible sequence do not overlap. Previously we eliminated strings because they contained a particular substring which was forbidden; now we eliminate strings on the basis of containing several forbidden substrings in a combination so that the full parameter range $(0, \pi)$ is accounted for. To give the simplest example, the string $+-$ can only appear in some admissible sequence for $0 < \theta < \frac{\pi}{2}$ and $++$ only for $\frac{\pi}{2} < \theta < \pi$, so any string containing both $+-$ and $++$ as substrings must be ruled out. To put together such coordinated combinations of strings, we determined for each string of length 2, 3 or 4 the range within which the string can appear in some admissible sequence. This information was used to assemble Table 4.11; we list strings and their ranges for θ where each may appear in some admissible sequence, but we do not include any string where the corresponding range for θ is no smaller than the intersection of the ranges for some of its substrings. As an example, the string $--0+$ is not listed because its range, $\frac{2\pi}{3} < \theta < \pi$, is the intersection of the ranges of its substrings $--$ and $-0+$. The ranges given in Table 4.11 have been deduced from our work in Chapter 3 on strings that appear within admissible sequences; all isolated single parameter values and interval end-points have been suppressed on the basis that we are ignoring instances of admissibility that do not persist for more than an isolated value of θ .

Careful consideration of all possible combinations of the strings from Table 4.11 gives a minimal list of tests to disqualify strings (throughout the entire range $(0, \pi)$) on the basis of substrings they contain of lengths 2, 3 and 4. This list of tests appears in Table 4.12; the notation employed there uses an \mathbf{x} symbol to represent a non-zero digit and, as usual, a bar indicates negation, so that the string $0\bar{\mathbf{x}}\mathbf{x}0$ stands for both $0+-0$ and $0--0$. This form of display matches the tests performed by the computer code where it is just as easy, and more efficient, to look for a substring $\mathbf{x}0\mathbf{x}$ consisting of two equal non-zero digits enclosing a zero, than separately to look for the two substrings $+0+$ and $-0-$.

| String Length | String | Ranges for θ |
|---------------|--------|---|
| 2 | -- | $(\frac{\pi}{2}, \pi)$ |
| | -+ | $(0, \frac{\pi}{2})$ |
| | +- | $(0, \frac{\pi}{2})$ |
| | ++ | $(\frac{\pi}{2}, \pi)$ |
| 3 | -0- | no value of θ |
| | -0+ | $(0, \frac{\pi}{3}), (\frac{2\pi}{3}, \pi)$ |
| | 0-0 | $(\frac{\pi}{3}, \frac{2\pi}{3})$ |
| | 0+0 | $(\frac{\pi}{3}, \frac{2\pi}{3})$ |
| | +0- | $(0, \frac{\pi}{3}), (\frac{2\pi}{3}, \pi)$ |
| | +0+ | no value of θ |
| | | |
| 4 | -00- | $(\frac{\pi}{3}, \frac{\pi}{2}), (\frac{3\pi}{4}, \pi)$ |
| | -00+ | $(0, \frac{\pi}{4}), (\frac{\pi}{2}, \frac{2\pi}{3})$ |
| | 0--0 | $(\frac{2\pi}{3}, \pi)$ |
| | 0-+0 | $(0, \frac{\pi}{3})$ |
| | 0+-0 | $(0, \frac{\pi}{3})$ |
| | 0++0 | $(\frac{2\pi}{3}, \pi)$ |
| | +00- | $(0, \frac{\pi}{4}), (\frac{\pi}{2}, \frac{2\pi}{3})$ |
| | +00+ | $(\frac{\pi}{3}, \frac{\pi}{2}), (\frac{3\pi}{4}, \pi)$ |
| | | |

Table 4.11: Ranges for θ in which strings of lengths 2 to 4 appear within some admissible sequence. (The ranges for strings not listed are simply intersections of the ranges of some of their substrings that do appear in the list.)

The basis for the disqualification process could be extended to encompass strings of greater length, but the practical difficulties increase because as more strings are introduced the number of tests based on combinations of strings grows rapidly. For instance, the extra strings and their ranges when we move up to length 5 are

| | | | |
|-----------------|---|-----------------|-----------------------------------|
| $\bar{x}xx00$: | $(\frac{\pi}{4}, \frac{\pi}{2})$ | $0\bar{x}xx0$: | no value of θ |
| $xx\bar{x}00$: | $(\frac{\pi}{2}, \frac{3\pi}{4})$ | $0xx\bar{x}0$: | no value of θ |
| $\bar{x}x000$: | $(0, \frac{2\pi}{5})$ | $00\bar{x}xx$: | $(\frac{\pi}{4}, \frac{\pi}{2})$ |
| $xx000$: | $(\frac{3\pi}{5}, \pi)$ | $00xx\bar{x}$: | $(\frac{\pi}{2}, \frac{3\pi}{4})$ |
| $x000x$: | no value of θ | $000\bar{x}x$: | $(0, \frac{2\pi}{5})$ |
| $\bar{x}000x$: | $(0, \frac{\pi}{5}), (\frac{4\pi}{5}, \pi)$ | $000x\bar{x}$: | $(\frac{3\pi}{5}, \pi)$ |

and considerable effort would be required to list a minimal collection of tests on the basis

| Lengths Under Consideration | Forbidden Combinations of Strings |
|-----------------------------|--|
| 2 | $\bar{x}x \ \&\& \ xx$ |
| 3 | $x0x$ $\bar{x}0x \ \&\& \ 0x0$ |
| 4 | $0\bar{x}x0 \ \&\& \ 0xx0$ $0\bar{x}x0 \ \&\& \ x00x$ $\bar{x}00x \ \&\& \ 0xx0$ $\bar{x}00x \ \&\& \ x00x$ |
| 3, 4 | $0x0 \ \&\& \ 0\bar{x}x0$ $0x0 \ \&\& \ 0xx0$ |
| 2, 3, 4 | $\bar{x}x \ \&\& \ \bar{x}0x \ \&\& \ x00x$ $\bar{x}x \ \&\& \ 0x0 \ \&\& \ \bar{x}00x$ $xx \ \&\& \ \bar{x}0x \ \&\& \ \bar{x}00x$ $xx \ \&\& \ 0x0 \ \&\& \ x00x$ |

Table 4.12: The tests used to eliminate strings which contain combinations of substrings of length 2, 3 or 4 whose ranges (from Table 4.11) are mutually exclusive. The symbol "&&" denotes logical "and"; the generic symbol x assumes the value + or -, independently in each string.

of combinations of strings of lengths 2, 3, 4 and 5.

The remainder of the implementation involves matters appearing earlier in the chapter so we do not repeat them here. It is relevant to mention that the method of elimination based on a cycled version of the string being disqualified, presented in Section 4.2.2, is an essential component of the algorithm for the comprehensive search because of the high computational cost now associated with checking whether a string ever generates an admissible periodic sequence. Our program enabled us to determine *all* admissible periodic sequences up to length 20, and the output, tabulated by increasing string length, appears as Table 3 in Volume II.

From the results of our comprehensive search, we have selected, for Table 4.13, those strings which generate an admissible periodic sequence but that do not belong to any of the families treated in our analytical investigations in Chapters 5 to 9 of this thesis, in effect those strings whose properties are still wholly unknown to us. (So that the list is as concise

as possible, we have suppressed all strings which may be obtained from those presented by negating all digits, or negating alternate digits; these may be deduced by means of Proposition 2.4.) The corresponding intervals of admissibility are recorded, together with their rational equivalents where these were found. We remark that, looking only as far as length 20, the comprehensive search uncovered several strings with such short intervals of admissibility that they did not appear amongst the results of our searches at fixed parameter values; these strings are indicated by an asterisk in Table 4.13 and would not otherwise have been uncovered by any of the avenues of investigation pursued in this thesis.

We conclude by suggesting that the comprehensive search strategy is the most valuable part of the computational investigation to pursue further, and it would be extremely interesting to generate a complete list of the admissible periodic sequences up to a string length of one or two hundred (i.e. *all* that are admissible *somewhere* within the range 0 to π). However, the computing techniques necessary for this task go beyond the methods we have described; the particular feature of the problem limiting us to a maximum length of 20 is the fast growth rate of the number of strings that remain after the elimination phase : of course, an elimination based on identifying forbidden combinations of substrings does not rule out nearly as many strings as was the case for the fixed parameter value searches.

| Period | Generating String | Interval of Admissibility |
|--------|-------------------------------------|---|
| 9 | ++++00-00 | (1.74532925, 1.98337054) = $(\frac{5\pi}{9}, -)$ |
| | +++++0--0 | (2.22579076, 2.44346095) = $(-, \frac{7\pi}{9})$ |
| 12 | +---+--00+00 | (1.21075578, 1.28751985) |
| 13 | + - 0 + - 00000000 | (0.87495662, 0.880896909) |
| | +++++00000000 | (2.26663603, 2.28558262) |
| 14 | + - + - 0000000000 | (1.18878695, 1.19963216) |
| 15 | ++++00-00+00-00 | (1.88495559, 1.95382806) = $(\frac{3\pi}{5}, -)$ |
| 16 | +00+00+0000-0000 | (1.3384859, 1.35427013) |
| 17 | + - 0 + - 0 + - 0 + - 00 + - 00 | (0.564813763, 0.675508053) |
| | * + - 00 + - 000000000000 | (0.691735065, 0.692648183) |
| | * + - + - 0 - + 000000+ - 0 | (0.722145541, 0.72524333) |
| | + - + - + 000000000000 | (0.829312926, 0.835542105) |
| | * + + + + + + 0000000000 | (1.90552987, 1.90972602) |
| | * + + 0 - - 000000000000 | (2.30195493, 2.30605055) |
| | +++++0--00--00--0 | (2.41634932, 2.4660846) |
| 19 | + - + - 0000000000000000 | (1.21710252, 1.22060548) |
| | * + - + - 00 - 0000000000+00 | (1.3056328, 1.30748473) |
| | ++++00-0000-0000-00 | (1.72830849, 1.83410793) |
| | * +00-0000000000000000 | (1.92449013, 1.92559133) |
| 20 | + - + - + - + - + - + - + - 0 + - 0 | (0.785398163, 0.831936667) = $(\frac{\pi}{4}, -)$ |
| | * + - 0 + - 0000000000000000 | (0.887865432, 0.889006322) |

Table 4.13: An extract from the comprehensive search results : the admissible periodic sequences of period up to 20 which do not belong to one of the infinite families identified and investigated subsequently in this thesis.

Chapter Summary

- An efficient computational strategy is presented that executes a search for the periodic sequences admissible at any specified value of the parameter θ .
- Important features of the algorithm are :
 - Early exclusion of the vast majority of potential generating strings on the basis of embedded forbidden substrings.
 - Automated computation of the geometric intersections associated with the criterion given in Chapter 3.
 - Digit-by-digit assembly of generating strings via a computation organised as a walk over a ternary tree whose nodes are the generating strings.
 - A fast implementation of the admissibility criterion established in Chapter 2.

Computational runtime costs are of order $O(1.022^N)$, which compares well with the runtime cost $O(3^N)$ for the naive algorithm that exhaustively searches amongst the full collection of conceivable generating strings of length N .

- Intervals of admissibility for the sequences are determined by a bisection algorithm.
- Generating strings for admissible periodic sequences have been tabulated for string length N up to 200 and where the parameter θ is advanced by steps of 0.01 across the whole parameter range 0 to π . To our knowledge this is the first substantial assemblage of computational data on admissible periodic sequences. Subsequent theoretical investigations are motivated by these computer search results; a first observation is a clustering of sequences around $\theta = 0, \pi, \frac{\pi}{2}, \frac{\pi}{3}$ and $\frac{2\pi}{3}$.
- Confirmatory calculations using rational arithmetic at parameter value $a = 2 \cos \theta = 0.5$ match the floating-point results and validate the numerical stability of our methods.
- Also tabulated are the results of a search that is comprehensive in terms of providing for admissibility *anywhere* within the θ range from 0 to π ; listed are *all* strings of length $N \leq 20$ that generate an admissible periodic sequence within some interval of θ values.

Chapter 5

Some Infinite Families of Admissible Periodic Sequences

The Chua-Lin inequalities provide a very powerful tool for the investigation of admissible periodic sequences. As a first application we produce some specific solutions to these inequalities. We consider three types of string pattern and produce several infinite families of admissible periodic sequences. The first two types involve admissibility close to $\frac{\pi}{2}$ and zero respectively, and by comparison with our subsequent investigations, these cases are relatively amenable and we can be completely definite about their intervals of admissibility. Our third type, which produces families of periodic sequences admissible close to $\frac{\pi}{4}$, requires a deeper analysis and the tools developed there would be effective beyond this specific context.

5.1 Admissible periodic sequences generated by the strings $+0 \dots 0$

The first type of periodic sequence we investigate are those generated by repeating a string $+0 \dots 0$, that is a single $+$ digit followed by a run of 0 digits. Sequences of this form are good candidates to take as a starting point for analytical investigations for the following reasons :

- (a) When the string is $+0 \dots 0$, the Chua-Lin inequalities assume a very simple form, since each of the functions f_r contains only a single cosine term with non-zero coefficient.
- (b) The computer searches of Chapter 4 found numerous instances of admissible sequences generated by strings $+0 \dots 0$. The sequences of this kind uncovered by computer exploration have clearly identifiable patterns in terms both of the lengths of the strings which generate them, and of their computed intervals of admissibility.

Table 5.1, which has been assembled from the lists in Table 1 of Volume II for parameter values either side of $\frac{\pi}{2}$, exhibits strings $+0 \dots 0$ with length ≤ 20 which were found to generate admissible periodic sequences. Two families of strings can be distinguished here, those of lengths $N = 1, 5, 9, 13$ and 17 which generate sequences admissible in an interval of θ values having $\frac{\pi}{2}$ as *left* end-point, and those of lengths $N = 3, 7, 11, 15$ and 19 which generate sequences admissible in an interval having $\frac{\pi}{2}$ as *right* end-point. Computer searches at fixed θ values either side of $\frac{\pi}{2}$ show these families persist for corresponding string lengths up to the maximum length that could be encompassed by the computer searches, so one might conjecture that these families extend indefinitely.

The purpose of this section is to show that indeed the following two infinite families of strings of the type $+0 \dots 0$ do generate admissible periodic sequences and for the range of θ indicated.

1. Strings $+0 \dots 0$ of lengths $N = 1, 5, 9, 13, \dots$ for the interval of θ values $\left(\frac{\pi}{2}, \frac{(N+1)\pi}{2N}\right)$.
2. Strings $+0 \dots 0$ of lengths $N = 3, 7, 11, 15, \dots$ for the interval of θ values $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$.

Before presenting the series of results that will establish what we claim generally for these families, we will look in detail at one specific such string. As an example it illustrates the techniques we will use to work with the Chua-Lin inequalities, but allows us to be free from complications necessary when we present the general argument.

| String Length (N) | Generating String | Interval of Admissibility |
|--------------------------|---------------------|------------------------------------|
| 1 | + | $(\frac{\pi}{2}, \pi)$ |
| 3 | +00 | $(\frac{\pi}{3}, \frac{\pi}{2})$ |
| 5 | +0000 | $(\frac{\pi}{2}, \frac{3\pi}{5})$ |
| 7 | +000000 | $(\frac{3\pi}{7}, \frac{\pi}{2})$ |
| 9 | +00000000 | $(\frac{\pi}{2}, \frac{5\pi}{9})$ |
| 11 | +0000000000 | $(\frac{5\pi}{11}, \frac{\pi}{2})$ |
| 13 | +000000000000 | $(\frac{\pi}{2}, \frac{7\pi}{13})$ |
| 15 | +00000000000000 | $(\frac{7\pi}{15}, \frac{\pi}{2})$ |
| 17 | +0000000000000000 | $(\frac{\pi}{2}, \frac{9\pi}{17})$ |
| 19 | +000000000000000000 | $(\frac{9\pi}{19}, \frac{\pi}{2})$ |

Table 5.1: Strings +0...0 which generate admissible periodic sequences, as found by the computer searches.

5.1.1 Example : Verifying the interval of admissibility when $N = 7$

Here, by way of an example, we will verify that the string +000000, of length $N = 7$, does indeed generate an admissible periodic sequence when θ lies between $\frac{3\pi}{7}$ and $\frac{\pi}{2}$, as the computer searches predict. In this instance, the functions f_r and g appearing in the Chua-Lin inequalities are

$$f_0(\theta) = \cos \frac{7}{2}\theta$$

$$f_1(\theta) = \cos \frac{5}{2}\theta$$

$$f_2(\theta) = \cos \frac{3}{2}\theta$$

$$f_3(\theta) = \cos \frac{1}{2}\theta$$

$$f_4(\theta) = \cos(-\frac{1}{2}\theta) = f_3(\theta)$$

$$f_5(\theta) = \cos(-\frac{3}{2}\theta) = f_2(\theta)$$

$$f_6(\theta) = \cos(-\frac{5}{2}\theta) = f_1(\theta)$$

and

$$g(\theta) = \sin \frac{7}{2} \theta \sin \theta.$$

As can be seen, only four of the seven functions f_r are distinct.

To verify the admissibility we need to establish the Chua-Lin inequalities over the range specified, i.e. that each function f_r lies between $-g$ and g for all θ in the interval $(\frac{3\pi}{7}, \frac{\pi}{2})$. Figure 5.1 shows the functions f_0, f_1, f_2 and f_3 , together with $\pm g$, plotted over the full range of θ from 0 to π . An interval of θ values where the Chua-Lin inequalities hold appears on the plot as a horizontal span over which all of the functions f_r lie within the envelope formed by the curves $\pm g$. Certainly in Figure 5.1 there is such an interval, located immediately to the left of $\theta = \frac{\pi}{2}$.

Figure 5.2 shows the same functions, but the plot is limited to the range $[\frac{3\pi}{7}, \frac{\pi}{2}]$ where we seek to demonstrate the admissibility of the string. All of f_0, f_1, f_2 and f_3 do lie between $-g$ and g as anticipated, but a significant feature for us that the plot indicates, is that f_1 dominates f_0, f_2 and f_3 in magnitude over the interval $(\frac{3\pi}{7}, \frac{\pi}{2})$. Establishing that a single f_r dominates over an interval will prove to be extremely useful, and is a theme that recurs frequently in the work associated with the Chua-Lin inequalities throughout this chapter. In the present context, if we demonstrate that one of the functions f_0, f_1, f_2 and f_3 dominates the others over the interval $(\frac{3\pi}{7}, \frac{\pi}{2})$, the issue of admissibility for this interval then depends only on that single inequality, and we can discard the six other Chua-Lin inequalities.

The arguments of the cosine functions in f_0, f_1, f_2 and f_3 are respectively $\frac{7}{2}\theta, \frac{5}{2}\theta, \frac{3}{2}\theta$ and $\frac{1}{2}\theta$, and the range of each as θ varies between $\frac{3\pi}{7}$ and $\frac{\pi}{2}$ is as follows :

$$f_0 : \frac{7}{2}\theta \text{ runs from } \frac{3\pi}{2} \text{ to } \frac{7\pi}{4},$$

$$f_1 : \frac{5}{2}\theta \text{ runs from } \frac{15\pi}{14} \text{ to } \frac{5\pi}{4},$$

$$f_2 : \frac{3}{2}\theta \text{ runs from } \frac{9\pi}{14} \text{ to } \frac{3\pi}{4},$$

$$f_3 : \frac{1}{2}\theta \text{ runs from } \frac{3\pi}{14} \text{ to } \frac{\pi}{4}.$$

These ranges are depicted graphically as angles in Figure 5.3 below. Notice that each range

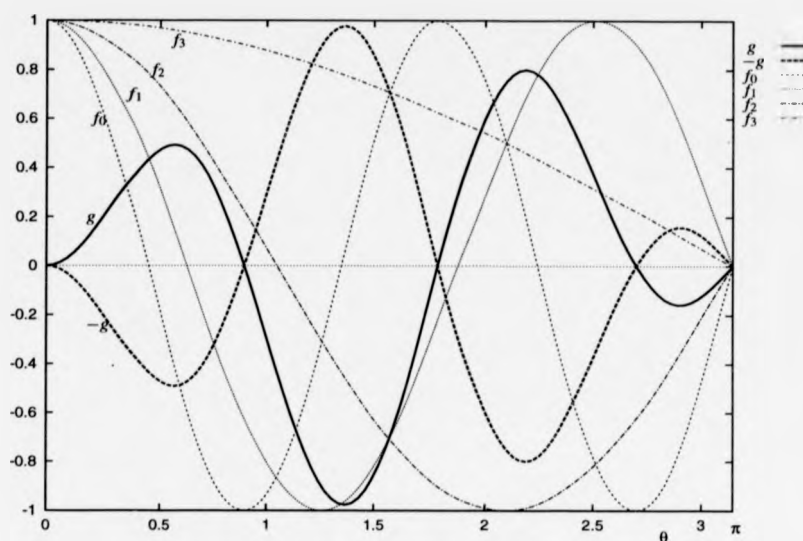


Figure 5.1: Plot of f_0, f_1, f_2, f_3 and $\pm g$ for the string $+000000$. All of f_0 to f_3 lie within the envelope of $\pm g$ for a short interval immediately to the left of $\frac{\pi}{2}$.

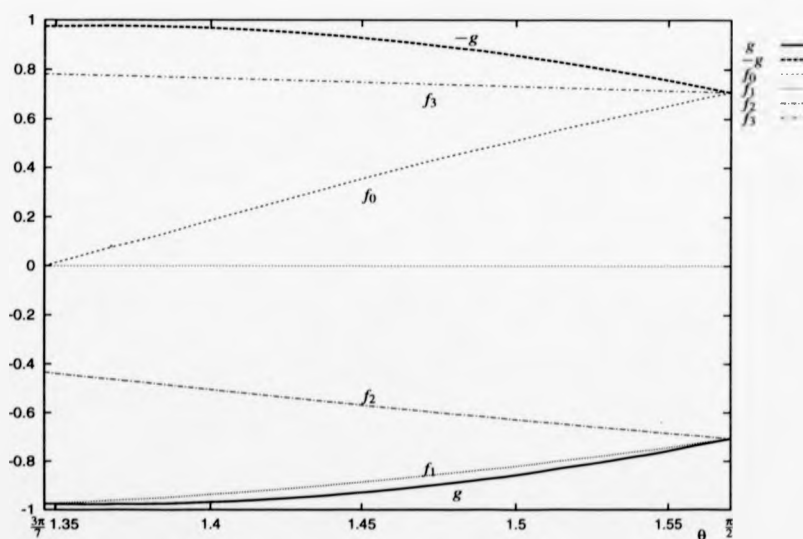


Figure 5.2: An enlargement of the plot above, limited to the range $\frac{3\pi}{7} \leq \theta \leq \frac{\pi}{2}$, where the string generates an admissible periodic sequence. Over this range f_1 dominates, in magnitude, the functions f_0, f_2 and f_3 .

is confined to a single quadrant, so that each function f_r either monotonically increases or decreases with θ when θ is restricted to the range $\frac{3\pi}{7}$ to $\frac{\pi}{2}$. Table 5.2 summarises the values of f_r at the two ends of the range, and whether the absolute value of the function increases or decreases as θ decreases from $\frac{\pi}{2}$ to $\frac{3\pi}{7}$.

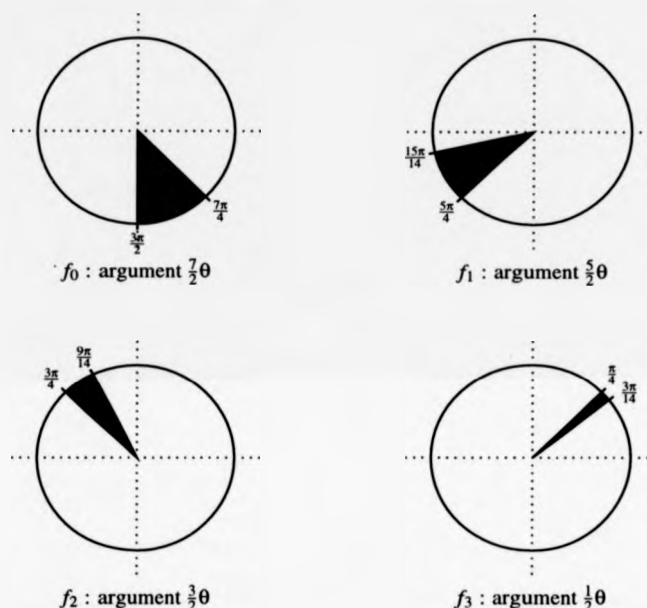


Figure 5.3: Angles of the arguments of the cosine functions in f_0 , f_1 , f_2 and f_3 for θ in the range $\frac{3\pi}{7}$ to $\frac{\pi}{2}$.

At the upper end of the range, $\theta = \frac{\pi}{2}$, the argument of each f_r is an odd multiple of $\frac{\pi}{4}$, so the four functions are identical in absolute value. (This can also be seen in Figure 5.2 where the graphs of the f_r come together at the right-hand end of the plot.) As θ decreases from $\frac{\pi}{2}$ to $\frac{3\pi}{7}$, so $|\cos \frac{5}{2}\theta|$ increases while both $|\cos \frac{7}{2}\theta|$ and $|\cos \frac{3}{2}\theta|$ decrease. Thus the ratios $|f_0/f_1|$ and $|f_2/f_1|$ fall from their initial value 1 as θ is reduced, and consequently f_1 dominates both f_0 and f_2 in absolute value.

The function f_3 cannot be treated so simply because $|\cos \frac{1}{2}\theta|$ increases when θ de-

| Function f_r | Values of f_r at the end-points of the range | | Trend of $ f_r $ as θ is decreased through its range |
|----------------|--|--------------------------|---|
| | $\theta = \frac{3\pi}{7}$ | $\theta = \frac{\pi}{2}$ | |
| f_0 | 0 | $\frac{1}{\sqrt{2}}$ | decreasing |
| f_1 | $\cos\left(\frac{15\pi}{14}\right)$ | $-\frac{1}{\sqrt{2}}$ | increasing |
| f_2 | $\cos\left(\frac{9\pi}{14}\right)$ | $\frac{1}{\sqrt{2}}$ | decreasing |
| f_3 | $\cos\left(\frac{3\pi}{14}\right)$ | $\frac{1}{\sqrt{2}}$ | increasing |

Table 5.2: Behaviour of the functions f_0 to f_3 when θ is in the range $(\frac{3\pi}{7}, \frac{\pi}{2})$

creases from $\frac{\pi}{2}$ to $\frac{3\pi}{7}$. But this cosine term has argument $\frac{1}{2}\theta$, by contrast with $\frac{5}{2}\theta$ in f_1 . As a consequence $|f_3/f_1|$ also falls as θ decreases from $\frac{\pi}{2}$ to $\frac{3\pi}{7}$ (the point is that $\cos\frac{5}{2}\theta$ and $\cos\frac{1}{2}\theta$ trace out congruent parts of the cosine curve but the latter does so at a slower rate).

Now that we have shown that f_1 dominates f_0 , f_2 and f_3 over the interval $(\frac{3\pi}{7}, \frac{\pi}{2})$, it remains to demonstrate that f_1 is enclosed between $-g$ and g . The function f_1 stays negative throughout the range (which is clear from the argument diagram in Figure 5.3). The sign of g is determined by the sign of the factor $\sin\frac{7}{2}\theta$, since $\sin\theta$ is positive for all $0 < \theta < \pi$. Again from Figure 5.3, it can be seen that $\sin\frac{7}{2}\theta$ is also negative for θ between $\frac{3\pi}{7}$ and $\frac{\pi}{2}$. As f_1 and g are both negative, it is certainly the case that $f_1 < -g$, so the outstanding task is to show that $f_1 > g$ for all θ between $\frac{3\pi}{7}$ and $\frac{\pi}{2}$. More specifically we need to verify that

$$\cos\frac{5}{2}\theta > \sin\frac{7}{2}\theta \sin\theta$$

when $\frac{3\pi}{7} < \theta < \frac{\pi}{2}$. If we express the product of sine terms additively as $\frac{1}{2}(\cos\frac{5}{2}\theta - \cos\frac{9}{2}\theta)$, and combine the common $\cos\frac{5}{2}\theta$ terms we find that we need to establish

$$\cos\frac{9}{2}\theta + \cos\frac{5}{2}\theta > 0.$$

Again, converting the sum to a product, we must show

$$2 \cos \frac{7}{2}\theta \cos \theta > 0,$$

which is true because both $\cos \frac{7}{2}\theta$ and $\cos \theta$ are positive for $\frac{3\pi}{7} < \theta < \frac{\pi}{2}$. We can thus conclude that the string $+000000$ generates a periodic sequence admissible for $\frac{3\pi}{7} < \theta < \frac{\pi}{2}$.

5.1.2 The admissibility of the infinite families of strings $+0 \dots 0$

The two propositions of this section develop the ideas of the example just considered to establish more generally the admissibility, over the ranges of θ previously conjectured, of the periodic sequences generated by the strings $+0 \dots 0$. As in the example, demonstrating the admissibility is done in two steps :

1. We show that over the range of θ under consideration, a single f_r dominates, in magnitude, the other functions.
2. We simplify the one crucial Chua-Lin inequality and thereby check that it is satisfied for the relevant θ values.

Proposition 5.1

The string $+0 \dots 0$ of length N , when $N = 3, 7, 11, \dots$, generates a periodic sequence admissible for all θ in the interval $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$.

Proof

The first $\frac{1}{2}(N+1)$ functions f_r are

$$\begin{aligned} f_0(\theta) &= \cos \frac{N}{2}\theta, \\ f_1(\theta) &= \cos \frac{N-2}{2}\theta, \\ &\vdots \\ f_{\frac{1}{2}(N-1)}(\theta) &= \cos \frac{1}{2}\theta, \end{aligned} \tag{5.1}$$

and they are all distinct. We need only consider these functions since the remaining f_r 's duplicate members of this list. As in the example of Section 5.1.1, we show that f_1 dominates in absolute value all other f_r 's over the interval $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$.

Because N is odd, the arguments $\frac{N}{2}\theta, (\frac{N}{2}-1)\theta, \dots, \frac{1}{2}\theta$, of the cosine functions appearing in the f_r 's, are all odd multiples of $\frac{\pi}{4}$ when $\theta = \frac{\pi}{2}$; thus each $|f_r(\frac{\pi}{2})| = \frac{1}{\sqrt{2}}$. As θ traverses its interval $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$, so the argument $\frac{N}{2}\theta$ traces out the largest angle, from $\frac{(N-1)\pi}{4}$ to $\frac{N\pi}{4}$, and this lies entirely within one quadrant. The angle traced out by any other argument also lies within one quadrant, because that angle is smaller and likewise finishes at an odd multiple of $\frac{\pi}{4}$. Thus as we imagine θ receding through its interval $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$, the absolute value of each f_r will either monotonically increase or decrease from $\frac{1}{\sqrt{2}}$.

To employ the technique that we used in the example, we need f_1 to increase in absolute value as θ decreases through its interval. The argument associated with f_1 is $\frac{N-2}{2}\theta$ and since N is an odd number of the form $4m-1$, the angle traced out lies either in the first quadrant (when $N = 3, 11, 19, \dots$) or the third quadrant (when $N = 7, 15, 23, \dots$). These angles are shown in the upper half of Figure 5.4; in both cases $|f_1|$ increases from $\frac{1}{\sqrt{2}}$ as θ decreases from $\frac{\pi}{2}$ to $\frac{(N-1)\pi}{2N}$. It follows then that $|f_r/f_1| < 1$ when f_r is one of the functions $f_2, f_3, \dots, f_{\frac{1}{2}(N-1)}$, because the absolute value of each of these functions either decreases as θ decreases from $\frac{\pi}{2}$ to $\frac{(N-1)\pi}{2N}$, or increases at a slower rate than f_1 .

To complete the demonstration of step 1 it remains to show that $|f_0/f_1| < 1$, which will be so if, in magnitude, f_0 decreases with θ . It does so because the relevant argument to consider is $\frac{N}{2}\theta$ for f_0 , which lies in either the second or fourth quadrant when $\frac{(N-1)\pi}{2N} < \theta < \frac{\pi}{2}$ (see Figure 5.4).

Now that we know f_1 dominates, it is relatively straightforward to verify step 2, i.e. f_1 lies between $-g$ and g . Referring once more to Figure 5.4, we can see that the sign of f_1 (i.e. the sign of $\cos \frac{N-2}{2}\theta$) is the same as the sign of g (determined by the sign of $\sin \frac{N}{2}\theta$). Since f_1 and $-g$ are of opposite signs, only the inequality involving f_1 and g need be considered. This inequality must be $f_1 < g$ when f_1 and g are positive, and $f_1 > g$ when f_1 and g are negative. Thus we must show

$$\cos \frac{N-2}{2}\theta < \sin \frac{N}{2}\theta \sin \theta \quad \text{when } N = 3, 11, 19, \dots,$$

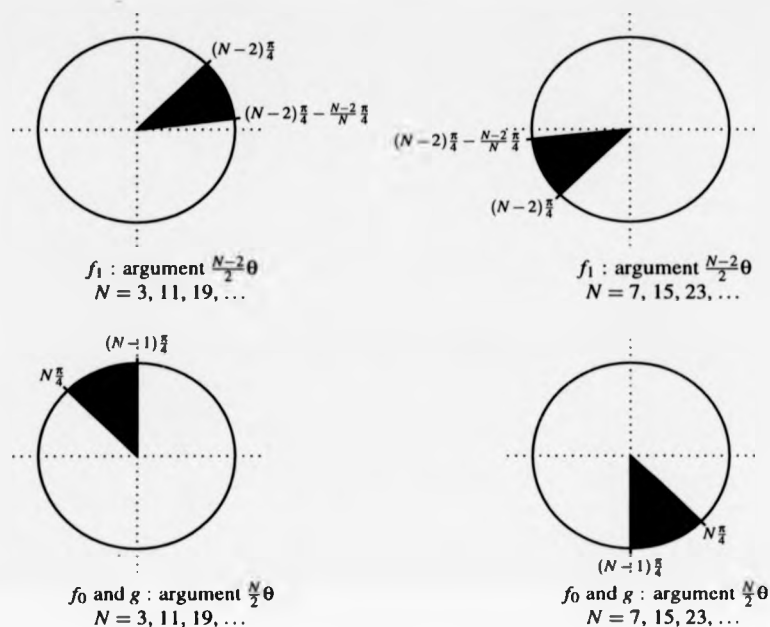


Figure 5.4: Angles of the arguments $\frac{N-2}{2}\theta$ (appearing in the cosine function in f_1) and $\frac{N}{2}\theta$ (appearing in the cosine function in f_0 and the sine function in g) for θ in the range $\frac{(N-1)\pi}{2N}$ to $\frac{\pi}{2}$.

and

$$\cos \frac{N-2}{2}\theta > \sin \frac{N}{2}\theta \sin \theta \quad \text{when } N = 7, 15, 23, \dots$$

Applying the same trigonometric simplification as in the example of Section 5.1.1 (i.e. essentially noting that $\cos \frac{N-2}{2}\theta - \sin \frac{N}{2}\theta \sin \theta = \cos \frac{N}{2}\theta \cos \theta$) our inequalities reduce to

$$\cos \frac{N}{2}\theta \cos \theta < 0 \quad \text{when } N = 3, 11, 19, \dots$$

and

$$\cos \frac{N}{2}\theta \cos \theta > 0 \quad \text{when } N = 7, 15, 23, \dots$$

To justify these it suffices to note the sign of $\cos \frac{N}{2}\theta$ in Figure 5.4. □

Proposition 5.2

The string $+0 \dots 0$ of length $N = 5, 9, 13, \dots$ generates a periodic sequence admissible for all θ in the interval $\left(\frac{\pi}{2}, \frac{(N+1)\pi}{2N}\right)$.

Proof

We can give an argument that closely follows the one we used to prove Proposition 5.1; in particular the definitions of the functions f_r and $\pm g$ are unchanged, and the strategy again is to show that f_1 dominates all other f_r 's over the given interval. However, both the interval now under consideration, and the values for N differ from the previous case, so some details need to be amended. The following points summarise the main parts of the proof:

1. The argument $\frac{N}{2}\theta$ varies from $\frac{N\pi}{4}$ to $\frac{(N+1)\pi}{4}$, and correspondingly all the other arguments vary within a single quadrant when $\frac{\pi}{2} < \theta < \frac{(N+1)\pi}{2N}$.
2. Each $|f_r|$ takes the value $\frac{1}{\sqrt{2}}$ when $\theta = \frac{\pi}{2}$. As θ increases from $\frac{\pi}{2}$ to $\frac{(N+1)\pi}{2N}$, so $|f_1|$ increases and $|f_r/f_1| < 1$ for $r = 2, 3, \dots, \frac{1}{2}(N-1)$; $|f_0|$ decreases and hence $|f_0/f_1| < 1$.
3. The argument $\frac{N-2}{2}\theta$ falls one quadrant in advance of $\frac{N-2}{2}\theta$ so $\cos \frac{N-2}{2}\theta$ and $\sin \frac{N}{2}\theta$ share a common sign when $\frac{\pi}{2} < \theta < \frac{(N+1)\pi}{2N}$, as do f_1 and g . The proof is completed by verifying that $f_1 > g$ when $N = 5, 13, 21, \dots$, and $f_1 < g$ when $N = 9, 17, 25, \dots$ for θ between $\frac{\pi}{2}$ and $\frac{(N+1)\pi}{2N}$, in the same manner as previously. \square

Remark on the Case $N = 1$

The string consisting of a single $+$ can be viewed as the case $N = 1$ of the family of strings $+0 \dots 0$. The formula for the interval of admissibility $\left(\frac{\pi}{2}, \frac{(N+1)\pi}{2N}\right)$ when $N = 1$ correctly specifies the interval $\left(\frac{\pi}{2}, \pi\right)$. However this string does not properly fit the framework of Proposition 5.2. But the Chua-Lin inequalities are simple for the string $+$, there is just one inequality which may be solved explicitly to find the values of θ for which the string

generates an admissible periodic sequence. The single Chua-Lin inequality is

$$-\sin \frac{1}{2}\theta \sin \theta < \cos \frac{1}{2}\theta < \sin \frac{1}{2}\theta \sin \theta. \quad (5.2)$$

Since all the sine and cosine terms are positive for all $0 < \theta < \pi$, the left-hand inequality holds uniformly. Writing $\sin \theta = 2 \sin \frac{1}{2}\theta \cos \frac{1}{2}\theta$, the right-hand inequality simplifies to

$$\sin^2 \frac{1}{2}\theta > \frac{1}{2}$$

which is satisfied precisely when $\frac{\pi}{2} < \theta < \pi$.

5.1.3 The question of admissibility elsewhere

We have obtained for each string $+0\dots 0$ of length the odd integer N an interval of θ values within which the periodic sequence generated by that string is admissible: the interval is $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$ when $N = 3, 7, 11, \dots$, and $\left(\frac{\pi}{2}, \frac{(N+1)\pi}{2N}\right)$ when $N = 5, 9, 13, \dots$. In fact, this interval is *maximal* in that it is not contained in some larger open interval throughout which the string generates an admissible periodic sequence.

This may be demonstrated by showing that to either side of the interval, the function f_1 leaves the envelope of $\pm g$. We discuss carefully the case where N is from the series of string lengths $3, 7, 11, \dots$, the other possibility is exactly similar; we are thus involved with the interval $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$. It was shown in the proof of Proposition 5.1 that f_1 and g share a common value at $\theta = \frac{\pi}{2}$. They also share a value at the left-hand end of the interval, $\frac{(N-1)\pi}{2N}$, as

$$f_1(\theta) = \cos \frac{N-2}{2}\theta, \quad (5.3)$$

and

$$\begin{aligned} g(\theta) &= \sin \frac{N}{2}\theta \sin \theta \\ &= \cos \frac{N-2}{2}\theta - \cos \frac{N}{2}\theta \cos \theta, \end{aligned} \quad (5.4)$$

and if $\theta = \frac{(N-1)\pi}{2N}$ then $f_1 = g$ because $\cos \frac{N}{2}\theta = \cos \frac{(N-1)\pi}{4} = 0$. We now compare the

values of the derivatives of f_1 and g at the ends of the interval to show that, at both, f_1 leaves the envelope of $\pm g$. Expressions for the derivatives of f_1 and g are

$$f_1'(\theta) = -\frac{N-2}{2} \sin \frac{N-2}{2} \theta, \quad (5.5)$$

$$g'(\theta) = \sin \frac{N}{2} \theta \cos \theta + \frac{N}{2} \cos \frac{N}{2} \theta \sin \theta. \quad (5.6)$$

Looking first at the left end-point $\theta = \frac{(N-1)\pi}{2N}$, $f_1 = g = \sin \frac{(N-1)\pi}{4} \sin \frac{(N-1)\pi}{2N}$ and

$$f_1' = -\frac{N-2}{2} \sin \frac{(N-1)\pi}{4} \cos \frac{(N-1)\pi}{2N},$$

$$g' = \sin \frac{(N-1)\pi}{4} \cos \frac{(N-1)\pi}{2N}.$$

When $N = 3, 11, \dots$, $f_1 = g > 0$, $f_1' < 0$ and $g' > 0$, whereas when $N = 7, 15, \dots$, $f_1 = g < 0$, $f_1' > 0$ and $g' < 0$; both possibilities mean that f_1 leaves the envelope of $\pm g$ immediately to the left of $\theta = \frac{(N-1)\pi}{2N}$.

At the right end-point $\theta = \frac{\pi}{2}$, $f_1 = g = \sin \frac{(N-1)\pi}{4} \cdot \frac{1}{\sqrt{2}}$ and

$$f_1' = -\frac{N-2}{2} \sin \frac{(N-1)\pi}{4} \cdot \frac{1}{\sqrt{2}},$$

$$g' = -\frac{N}{2} \sin \frac{(N-1)\pi}{4} \cdot \frac{1}{\sqrt{2}}.$$

When $N = 3, 11, \dots$, $f_1 = g > 0$, $g' < f_1' < 0$ whereas when $N = 7, 15, \dots$, $f_1 = g < 0$, $g' > f_1' > 0$; so again under both possibilities f_1 leaves the envelope of $\pm g$ immediately to the right of $\theta = \frac{\pi}{2}$.

Remark : The same conclusion can be reached by entirely different means, namely by using the result from Chapter 3 that specifies when the strings $+0 \dots 0+$ may appear within admissible sequences. A string $+0 \dots 0$ of length N may only generate an admissible periodic sequence at those θ values for which the string $+0 \dots 0+$ of length $(N+1)$ may appear within some admissible sequence. According to Proposition 3.1 this occurs precisely when θ belongs to one of the intervals $\left(\frac{2\pi}{N+2} \frac{ks_1-1}{s_1}, \frac{2\pi}{N+2} \frac{ks_2+1}{s_2} \right)$, where $1 \leq k \leq \lfloor \frac{N+1}{2} \rfloor$ and the even integers s_1, s_2 satisfy $ks_1 \equiv 1 \pmod{N+2}$ and $ks_2 \equiv -1 \pmod{N+2}$. The choice $k = \frac{1}{4}(N+1)$,

implying $s_1 = 2N$ and $s_2 = 4$, produces the interval $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$ which coincides with the interval where we have shown that the string $+0 \dots 0$ generates an admissible periodic sequence. So we find again that the periodic sequence generated by the string $+0 \dots 0$ will fail to be admissible immediately to either side of the interval $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$.

The preceding considerations suggest a wider question : "Can there be additional intervals, located elsewhere within the full range 0 to π for θ , throughout which the periodic sequence generated by a string $+0 \dots 0$ is admissible ?" Computational evidence suggests that the answer is no, i.e. such sequences have just a single interval of admissibility. The exhaustive search for sequences with period not exceeding 20 revealed for the strings $+0 \dots 0$ only the intervals already described. Of course, for string lengths $N > 20$, our searches are less comprehensive, investigating only a discrete set of θ values across the range 0 to π . Nevertheless the body of data thereby produced is large, and no additional instance of a periodic sequence generated by $+0 \dots 0$ admissible elsewhere has been noticed.

In fact because of the relatively simple composition of the strings $+0 \dots 0$ we are able to show that there are no *other* intervals of admissibility besides those already described. To do this requires showing that throughout the remainder of the range from 0 to π for θ , at least one of the functions f_r falls outside of the envelope of $\pm g$ (i.e. $|f_r(\theta)| \geq |g(\theta)|$).

To determine the points of intersection of the various f_r 's with g or $-g$ is a difficult task, as the corresponding trigonometric equations do not have easy analytic solutions. Instead, we simplify the task by decomposing the function g . Recall that $g(\theta) = \sin \frac{N}{2}\theta \sin \theta$, so g is the product of two sine waves, the slowly varying factor $\sin \theta$ modulates the rapidly varying factor $\sin \frac{N}{2}\theta$. Figure 5.5 is a plot of $\pm g$ where N is large; the limiting influence of the factor $\sin \theta$ on the extent of the envelope $\pm g$ is clear. Three of the functions f_r , namely $\cos \frac{1}{2}\theta$, $\cos \frac{3}{2}\theta$ and $\cos \frac{5}{2}\theta$, have been added to the diagram, and we see that for whole ranges of θ values, these functions lie well beyond either g or $-g$, to the extent that the slowly varying $\sin \theta$ factor in g can be used to exclude functions f_r from the envelope of $\pm g$.

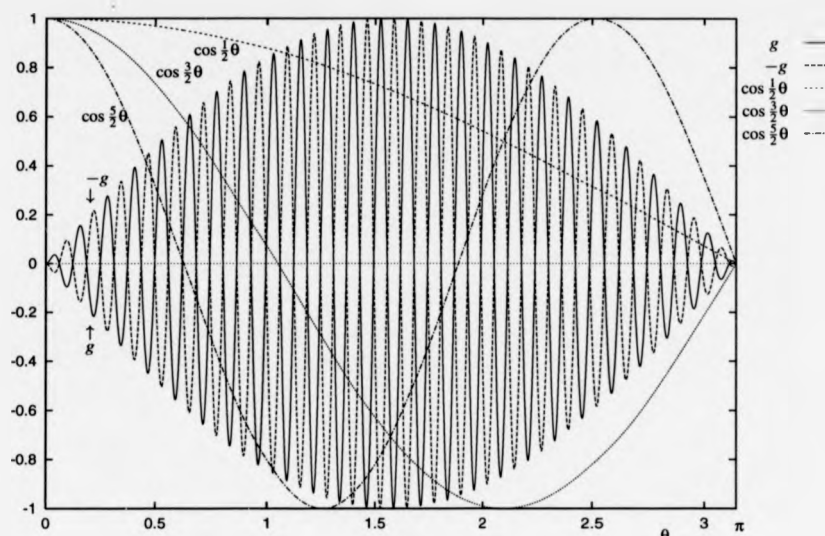


Figure 5.5: Plot of g and $-g$ when $N = 101$ showing the influence of the $\sin \theta$ factor on the shape of the envelope. The functions $\cos \frac{1}{2}\theta$, $\cos \frac{3}{2}\theta$ and $\cos \frac{5}{2}\theta$ have been superimposed; for large ranges of θ values they remain outside the envelope of $\pm \sin \theta$.

The function $\cos \frac{1}{2}\theta$ lies outside the envelope of $\pm \sin \theta$ for $0 \leq \theta \leq \frac{\pi}{3}$. This is because

$$\begin{aligned} \cos \frac{1}{2}\theta - \sin \theta &= \cos \frac{1}{2}\theta + \cos \left(\frac{\pi}{2} + \theta \right) \\ &= 2 \cos \frac{1}{2} \left(\frac{\pi}{2} + \frac{3}{2}\theta \right) \cos \frac{1}{2} \left(\frac{\pi}{2} + \frac{1}{2}\theta \right) \end{aligned}$$

and for $0 \leq \theta \leq \frac{\pi}{3}$ the ranges of the cosine function arguments $\frac{1}{2} \left(\frac{\pi}{2} + \frac{3}{2}\theta \right)$ and $\frac{1}{2} \left(\frac{\pi}{2} + \frac{1}{2}\theta \right)$ intervening here are $\left[\frac{\pi}{4}, \frac{\pi}{2} \right]$ and $\left[\frac{\pi}{4}, \frac{\pi}{3} \right]$ respectively, so that each cosine factor is positive. Thus $\cos \frac{1}{2}\theta \geq \sin \theta$ when $0 \leq \theta \leq \frac{\pi}{3}$. Consequently a string having the function $\cos \frac{1}{2}\theta$ as one of its f_r 's cannot generate an admissible periodic sequence when $0 \leq \theta \leq \frac{\pi}{3}$.

Applying this idea successively to the functions $\cos \frac{3}{2}\theta$, $\cos \frac{5}{2}\theta$, $\cos \frac{7}{2}\theta$, ..., we can generate the following table which lists the interval closest to $\frac{\pi}{2}$ where each function is outside the envelope of $\pm \sin \theta$.

| Function | Interval of Exclusion |
|---------------------------|-------------------------------------|
| $\cos \frac{1}{2}\theta$ | $[0, \frac{\pi}{3}]$ |
| $\cos \frac{3}{2}\theta$ | $[\frac{3\pi}{5}, \pi]$ |
| $\cos \frac{5}{2}\theta$ | $[\frac{\pi}{3}, \frac{3\pi}{7}]$ |
| $\cos \frac{7}{2}\theta$ | $[\frac{5\pi}{9}, \frac{3\pi}{5}]$ |
| $\cos \frac{9}{2}\theta$ | $[\frac{3\pi}{7}, \frac{5\pi}{11}]$ |
| $\cos \frac{11}{2}\theta$ | $[\frac{7\pi}{13}, \frac{5\pi}{9}]$ |

Note that the intervals for $\cos \frac{1}{2}\theta$ and $\cos \frac{5}{2}\theta$ are adjoining so, when both functions are present amongst the f_r 's, together they exclude the interval $[0, \frac{3\pi}{7}]$. We systematise the argument indicated here in the following lemma, and are thus able to show that any periodic sequence generated by a string $+0 \dots 0$ fails to be admissible over a large part of the range $0 < \theta < \pi$.

Lemma 5.3

The function $\cos \frac{k}{2}\theta$ is excluded from the envelope of $\pm \sin \theta$ for θ in $[\frac{(k+3)\pi}{2(k+2)}, \frac{(k-1)\pi}{2(k-2)}]$ when $k = 3, 7, 11, \dots$, and for θ in $[\frac{(k-3)\pi}{2(k-2)}, \frac{(k+1)\pi}{2(k+2)}]$ when $k = 5, 9, 13, \dots$

Proof

The function $\cos \frac{k}{2}\theta$ is excluded from the envelope of $\pm \sin \theta$ when either $\cos \frac{k}{2}\theta \geq \sin \theta$ or when $\cos \frac{k}{2}\theta \leq -\sin \theta$, because $\sin \theta \geq 0$ for $0 \leq \theta \leq \pi$. Noting that $\sin \theta = -\cos(\theta + \frac{\pi}{2})$ and expressing the sum of cosines multiplicatively, we have

$$\cos \frac{k}{2}\theta - \sin \theta = 2 \cos \frac{1}{2}((\frac{k}{2} + 1)\theta + \frac{\pi}{2}) \cos \frac{1}{2}((\frac{k}{2} - 1)\theta - \frac{\pi}{2}), \quad (5.7)$$

and the right-hand side of (5.7) is positive when the two factors $\cos \frac{1}{2}((\frac{k}{2} + 1)\theta + \frac{\pi}{2})$ and $\cos \frac{1}{2}((\frac{k}{2} - 1)\theta - \frac{\pi}{2})$ have the same sign. Likewise

$$\cos \frac{k}{2}\theta + \sin \theta = -2 \sin \frac{1}{2}((\frac{k}{2} + 1)\theta + \frac{\pi}{2}) \sin \frac{1}{2}((\frac{k}{2} - 1)\theta - \frac{\pi}{2}), \quad (5.8)$$

and this is negative when $\sin \frac{1}{2}((\frac{k}{2} + 1)\theta + \frac{\pi}{2})$ and $\sin \frac{1}{2}((\frac{k}{2} - 1)\theta - \frac{\pi}{2})$ have the same sign. Table 5.3 lists the ranges of the arguments $\frac{1}{2}((\frac{k}{2} + 1)\theta + \frac{\pi}{2})$ and $\frac{1}{2}((\frac{k}{2} - 1)\theta - \frac{\pi}{2})$, corre-

sponding to the intervals for θ that we are considering.

| Argument | Values at the End-points of the Interval $\left[\frac{(k+3)\pi}{2(k+2)}, \frac{(k-1)\pi}{2(k-2)}\right]$ | |
|--|--|--|
| | Left end-point | Right end-point |
| $\frac{1}{2} \left(\left(\frac{k}{2} + 1 \right) \theta + \frac{\pi}{2} \right)$ | $\frac{(k+5)\pi}{8}$ | $\frac{(k+5)\pi}{8} + \frac{1}{k-2} \frac{\pi}{2}$ |
| $\frac{1}{2} \left(\left(\frac{k}{2} - 1 \right) \theta - \frac{\pi}{2} \right)$ | $\frac{(k-3)\pi}{8} - \frac{1}{k+2} \frac{\pi}{2}$ | $\frac{(k-3)\pi}{8}$ |

| Argument | Values at the End-points of the Interval $\left[\frac{(k-3)\pi}{2(k-2)}, \frac{(k+1)\pi}{2(k+2)}\right]$ | |
|--|--|--|
| | Left end-point | Right end-point |
| $\frac{1}{2} \left(\left(\frac{k}{2} + 1 \right) \theta + \frac{\pi}{2} \right)$ | $\frac{(k+3)\pi}{8} - \frac{1}{k+2} \frac{\pi}{2}$ | $\frac{(k+3)\pi}{8}$ |
| $\frac{1}{2} \left(\left(\frac{k}{2} - 1 \right) \theta - \frac{\pi}{2} \right)$ | $\frac{(k-5)\pi}{8}$ | $\frac{(k-5)\pi}{8} + \frac{1}{k+2} \frac{\pi}{2}$ |

Table 5.3: Ranges of the arguments $\frac{1}{2} \left(\left(\frac{k}{2} + 1 \right) \theta + \frac{\pi}{2} \right)$ and $\frac{1}{2} \left(\left(\frac{k}{2} - 1 \right) \theta - \frac{\pi}{2} \right)$.

To justify the assertion of the lemma, we give a separate discussion for each of four possible cases for the index k .

(a) The case $k = 3, 11, 19, \dots$, so that we can write $k = 8l - 5$:

The interval to consider is $\left[\frac{(k+3)\pi}{2(k+2)}, \frac{(k-1)\pi}{2(k-2)}\right]$, and we use (5.8) to show that $\cos \frac{k}{2}\theta \leq -\sin \theta$ throughout this interval. The first argument varies from $l\pi$ to $l\pi + \frac{1}{8l-7} \frac{\pi}{2}$ and the second from $(l-1)\pi - \frac{1}{8l-3} \frac{\pi}{2}$ to $(l-1)\pi$, so that here we are involved with the third and fourth quadrants (odd l), or the first and second quadrants (even l). In either case the two factors on the right-hand side of (5.8) have the same sign.

(b) The case $k = 7, 15, 23, \dots$, so $k = 8l - 1$:

Again the interval is $\left[\frac{(k+3)\pi}{2(k+2)}, \frac{(k-1)\pi}{2(k-2)}\right]$, but we use (5.7) to show that $\cos \frac{k}{2}\theta \geq \sin \theta$. The first argument varies from $(l + \frac{1}{2})\pi$ to $(l + \frac{1}{2})\pi + \frac{1}{8l-3} \frac{\pi}{2}$ and the second from $(l - \frac{1}{2})\pi - \frac{1}{8l+1} \frac{\pi}{2}$ to $(l - \frac{1}{2})\pi$. Here we are involved with the first and fourth quadrants (odd l), or the second and third quadrants (even l). The two factors on the right-hand side of (5.7) have the same sign.

There remain

(c) The case $k = 5, 13, 21, \dots; k = 8l - 3$,

(d) The case $k = 9, 17, 25, \dots; k = 8l + 1$.

Both relate to the alternative interval $\left[\frac{(k-3)\pi}{2(k-2)}, \frac{(k+1)\pi}{2(k+2)}\right]$, but otherwise the details are entirely similar to those of cases (a) and (b). \square

All intervals for $k = 3, 7, 11, \dots$ are to the right of $\theta = \frac{\pi}{2}$ and all those for $k = 5, 9, 13, \dots$ are to the left. Any two successive intervals from the same family are adjacent; for example, if we consider $\left[\frac{(k+3)\pi}{2(k+2)}, \frac{(k-1)\pi}{2(k-2)}\right]$ and replace k by $(k+4)$ we obtain $\left[\frac{(k+7)\pi}{2(k+6)}, \frac{(k+3)\pi}{2(k+2)}\right]$, where the right end-point of the latter interval coincides with the left end-point of the former.

Proposition 5.4

The intervals $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$ when $N = 3, 7, 11, \dots$ and $\left(\frac{\pi}{2}, \frac{(N+1)\pi}{2N}\right)$ when $N = 5, 9, 13, \dots$ are the only intervals of admissibility for the periodic sequences generated by strings of the form $+0\dots 0$.

Proof

We look first at the case $N = 3, 7, 11, \dots$, for which the sequences have an interval of admissibility $\left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2}\right)$. Lemma 5.3 allows us to exclude the interval $\left[0, \frac{(N-1)\pi}{2N}\right]$, because it is the union $\left[0, \frac{\pi}{3}\right] \cup \left[\frac{\pi}{3}, \frac{3\pi}{7}\right] \cup \dots \cup \left[\frac{(N-5)\pi}{2(N-4)}, \frac{(N-1)\pi}{2N}\right]$ of excluded intervals. Similarly we can exclude the interval $\left[\frac{(N+3)\pi}{2(N+2)}, \pi\right] = \left[\frac{(N+3)\pi}{2(N+2)}, \frac{(N-1)\pi}{2(N-2)}\right] \cup \dots \cup \left[\frac{5\pi}{9}, \frac{3\pi}{5}\right] \cup \left[\frac{3\pi}{5}, \pi\right]$.

There remains a short interval, $\left[\frac{\pi}{2}, \frac{(N+3)\pi}{2(N+2)}\right]$, in which we must show that the sequences are not admissible. This interval is contained in the slightly longer interval $\left[\frac{\pi}{2}, \frac{(N+1)\pi}{2N}\right]$ and it will prove more convenient to work with this latter interval. This interval cannot be excluded by our argument for Lemma 5.3, based on $\sin \theta$ as a factor of g , rather we need to consider the second factor $\sin \frac{N}{2}\theta$. We show that one of the f_r 's is excluded from the envelope of $\pm \sin \frac{N}{2}\theta$ when θ is between $\frac{\pi}{2}$ and $\frac{(N+1)\pi}{2N}$.

As θ increases from $\frac{\pi}{2}$ to $\frac{(N+1)\pi}{2N}$, the argument $\frac{N}{2}\theta$ increases from $\frac{N\pi}{4}$ to $\frac{(N+1)\pi}{4}$. Writing

$N = 4m - 1$, $m = 1, 2, 3, \dots$, we find that the argument $\frac{N}{2}\theta$ increases from $m\pi - \frac{\pi}{4}$ to $m\pi$. Thus $|\sin \frac{N}{2}\theta|$ decreases monotonically from $\frac{1}{\sqrt{2}}$ to 0, while $|\cos \frac{N}{2}\theta|$ increases monotonically from $\frac{1}{\sqrt{2}}$ to 1. Consequently $|\cos \frac{N}{2}\theta| > |\sin \frac{N}{2}\theta|$ for $\frac{\pi}{2} < \theta \leq \frac{(N+1)\pi}{2N}$, but then f_0 is excluded from the envelope of $\pm g$.

The second case, when $N = 5, 9, 13, \dots$, may be treated by an entirely analogous argument; essentially we compare $|\cos \frac{N}{2}\theta|$ with $|\sin \frac{N}{2}\theta|$ over the interval $[\frac{(N-1)\pi}{2N}, \frac{\pi}{2}]$. \square

Thus we have determined exhaustively the intervals of admissibility for the periodic sequences generated by the strings $+0\dots 0$, when the string length N is odd. In fact this accounts for all admissible periodic sequences generated by any string $+0\dots 0$; there can be none for even N because we showed in Proposition 3.1 of Chapter 3 that a string $+0\dots 0+$ of odd length can never appear within an admissible sequence.

5.1.4 Related families of admissible sequences

Now that we have established the existence of a family of admissible periodic sequences, the symmetry results from Section 2.4 of Chapter 2 can be employed to produce some related infinite families of admissible periodic sequences.

A first result of this kind follows by negating each digit; we see from Proposition 2.4 that periodic sequences generated by the strings $-0\dots 0$ have the same intervals of admissibility as those generated by the strings $+0\dots 0$ previously considered. We obtain a new and more significant family of admissible periodic sequences by negating alternate digits. Consider the periodic sequence generated by the string $+00$, which we have shown is admissible for θ between $\frac{\pi}{3}$ and $\frac{\pi}{2}$. The string $+00+00$ generates the identical periodic sequence. But negating every other digit, we produce the string $+00-00$, and this generates a periodic sequence which is admissible for θ in the interval $(\pi - \frac{\pi}{2}, \pi - \frac{\pi}{3}) = (\frac{\pi}{2}, \frac{2\pi}{3})$. In this way we can create two further infinite families of periodic sequences. The first few members of each are listed in Table 5.4.

| String Length (N) | Generating String | Interval of Admissibility |
|--------------------------|---------------------------|-----------------------------------|
| 2 | +- | $(0, \frac{\pi}{2})$ |
| 10 | +0000-0000 | $(\frac{2\pi}{5}, \frac{\pi}{2})$ |
| 18 | +00000000-00000000 | $(\frac{4\pi}{9}, \frac{\pi}{2})$ |

| String Length (N) | Generating String | Interval of Admissibility |
|--------------------------|-------------------------------|------------------------------------|
| 6 | +00-00 | $(\frac{\pi}{2}, \frac{2\pi}{3})$ |
| 14 | +000000-000000 | $(\frac{\pi}{2}, \frac{4\pi}{7})$ |
| 22 | +0000000000-0000000000 | $(\frac{\pi}{2}, \frac{6\pi}{11})$ |

Table 5.4: The first three generating strings of each of the two additional infinite families of admissible periodic sequences arising from strings $+0 \dots 0$ by negating alternate digits.

The intervals of admissibility here are easily determined in terms of the intervals of the sequences that give rise to them. The interval of admissibility for the periodic sequence generated by the string $+0 \dots 0-0 \dots 0$ of length N is $(\frac{(N-2)\pi}{2N}, \frac{\pi}{2})$ when $N = 2, 10, 18, \dots$, and $(\frac{\pi}{2}, \frac{(N+2)\pi}{2N})$ when $N = 6, 14, 22, \dots$.

To summarise this section, we have derived full information on the range of admissibility for several infinite families of periodic sequences. A common feature is that $\frac{\pi}{2}$ is always an end-point of the interval of admissibility. We return to this in Appendix A, where we consider *all* the periodic sequences having an interval of admissibility with $\frac{\pi}{2}$ as an end-point.

5.2 Admissible periodic sequences generated by the strings $+-0 \dots 0$

The techniques introduced above for demonstrating the admissibility of sequences generated by the strings $+0 \dots 0$ can be used to investigate other families of periodic sequences.

In this section we will study the periodic sequences generated by the strings $+ - 0 \dots 0$. (Davies (1992) has also investigated the family of strings $+ - 0 \dots 0$, but by quite different methods to those we employ.) As previously, specific information obtained as a result of our computer searches suggests a general result about the intervals of admissibility. Strings of the type now considered and with length $N \leq 10$ are listed in Table 5.5, together with the associated intervals of admissibility.

| String Length (N) | Generating String | Interval of Admissibility |
|--------------------------|-------------------|---------------------------|
| 2 | $+ -$ | $(0, \frac{\pi}{2})$ |
| 3 | $+ - 0$ | $(0, \frac{\pi}{3})$ |
| 4 | $+ - 00$ | $(0, \frac{\pi}{4})$ |
| 5 | $+ - 000$ | $(0, \frac{\pi}{5})$ |
| 6 | $+ - 0000$ | $(0, \frac{\pi}{6})$ |
| 7 | $+ - 00000$ | $(0, \frac{\pi}{7})$ |
| 8 | $+ - 000000$ | $(0, \frac{\pi}{8})$ |
| 9 | $+ - 0000000$ | $(0, \frac{\pi}{9})$ |
| 10 | $+ - 00000000$ | $(0, \frac{\pi}{10})$ |

Table 5.5: Computer determined intervals of admissibility for periodic sequences generated by the strings $+ - 0 \dots 0$.

The clear pattern for the intervals of admissibility emerging here persists for greater string lengths throughout the data obtained from the computer searches. We shall show that a sequence of this type with period N is admissible for θ in the interval $(0, \frac{\pi}{N})$, and not throughout any open interval elsewhere. The functions f_r in the Chua-Lin inequalities are more complicated than previously, each is now the difference of two cosine terms,

$$f_r(\theta) = \cos \frac{N-2r}{2} \theta - \cos \frac{N-2(r+1)}{2} \theta \quad \text{for } r = 0, 1, \dots, N-1. \quad (5.9)$$

It will prove advantageous to write each f_r as the product of two sine terms,

$$f_r(\theta) = -2 \sin \left(\frac{1}{2}(N-1) - r \right) \theta \sin \frac{1}{2} \theta$$

because g may be factored as

$$\begin{aligned} g(\theta) &= \sin \frac{N}{2} \theta \sin \theta \\ &= 2 \sin \frac{N}{2} \theta \sin \frac{1}{2} \theta \cos \frac{1}{2} \theta, \end{aligned}$$

so that $|f_r(\theta)| < |g(\theta)|$ is equivalent to $|\sin(\frac{1}{2}(N-1)-r)\theta| < |\sin \frac{N}{2} \theta \cos \frac{1}{2} \theta|$. At this point it is convenient to define

$$\tilde{f}_r(\theta) = \sin(\frac{1}{2}(N-1)-r)\theta \quad (5.10)$$

$$\tilde{g}(\theta) = \sin \frac{N}{2} \theta \cos \frac{1}{2} \theta, \quad (5.11)$$

and the Chua-Lin inequalities for the sequences under consideration become $|\tilde{f}_r(\theta)| < |\tilde{g}(\theta)|$.

Note that there is some duplication (up to a sign) amongst the \tilde{f}_r 's, a situation that occurred before for the strings considered in Section 5.1. Dependent on whether N is odd or even, the following sets of functions can be taken to represent the complete collection of \tilde{f}_r 's:

$$\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{\frac{1}{2}(N-1)} \quad \text{when } N \text{ is odd,} \quad (5.12)$$

$$\tilde{f}_0, \tilde{f}_1, \dots, \tilde{f}_{\frac{1}{2}(N-2)} \quad \text{when } N \text{ is even.} \quad (5.13)$$

In the work that follows reference to "the collection of \tilde{f}_r 's" signifies the appropriate choice of function set described here.

The functions \tilde{f}_r and \tilde{g} are similar in character to the functions we worked with in Section 5.1 when investigating the periodic sequences generated by the strings $+0\dots 0$, and correspondingly the techniques developed there may be applied to the present case. A pleasant feature though is that the periodic sequences considered here, those generated by the strings $+ - 0\dots 0$, can be treated more uniformly, separate arguments for several series of cases are not needed. The results that follow specify precisely the range for θ when a periodic sequence generated by a string $+ - 0\dots 0$ is admissible.

Proposition 5.5

The periodic sequence generated by the string $+ - 0 \dots 0$ of length N is admissible for all θ in the interval $(0, \frac{\pi}{N})$.

Proof

We work with the collection of \bar{f}_r 's, (5.12) or (5.13), as described. Note first that \bar{f}_0 dominates all other \bar{f}_r 's for $0 < \theta < \frac{\pi}{N}$. This is because the argument $\frac{N-1-2r}{2}\theta$ of the sine function in \bar{f}_r increases from 0 to $\frac{N-1-2r}{2}\frac{\pi}{2}$ as θ traverses its range. All the arguments remain in the first quadrant, where the sine function is positive and increasing, so $\bar{f}_0(\theta) > \bar{f}_r(\theta) > 0$ when \bar{f}_r is other than \bar{f}_0 .

The second stage of the proof is to show that $|\bar{f}_0(\theta)| < |\bar{g}(\theta)|$ for $0 < \theta < \frac{\pi}{N}$. It suffices to show that $\bar{f}_0(\theta) < \bar{g}(\theta)$, because \bar{f}_0 and \bar{g} are positive when θ is between 0 and $\frac{\pi}{N}$. Now

$$\begin{aligned}\bar{g}(\theta) - \bar{f}_0(\theta) &= \sin \frac{N}{2}\theta \cos \frac{1}{2}\theta - \sin \frac{N-1}{2}\theta \\ &= \cos \frac{N}{2}\theta \sin \frac{1}{2}\theta,\end{aligned}$$

and this is positive since both $\cos \frac{N}{2}\theta$ and $\sin \frac{1}{2}\theta$ are for $0 < \theta < \frac{\pi}{N}$. □

Proposition 5.6

The interval $(0, \frac{\pi}{N})$ is the only interval of admissibility for the periodic sequence generated by the string $+ - 0 \dots 0$ of length N .

Proof

The method of proof is similar to that developed in Section 5.1, where we showed that sequences generated by strings $+ 0 \dots 0$ only have one interval of admissibility. We consider each function \bar{f}_r in turn and show that it rules out some portion of the full range $(0, \pi)$ of θ values.

We show first that $\sin \frac{k}{2}\theta$ is excluded from the envelope of $\pm \cos \frac{1}{2}\theta$ for θ in $[\frac{\pi}{2}, \pi]$ when $k = 1$, and for θ in $[\frac{\pi}{k+1}, \frac{\pi}{k-1}]$ when $k = 2, 3, 4, \dots$. Replacing $\cos \frac{1}{2}\theta$ by $\sin(\frac{\pi}{2} - \frac{1}{2}\theta)$ and

converting the difference of two sines to a product, we have

$$\sin \frac{k}{2}\theta - \cos \frac{1}{2}\theta = 2 \sin \left(\frac{k+1}{4}\theta - \frac{\pi}{4} \right) \cos \left(\frac{k-1}{4}\theta + \frac{\pi}{4} \right). \quad (5.14)$$

For θ in the interval $[\frac{\pi}{k+1}, \frac{\pi}{k-1}]$ the range of $\frac{k+1}{4}\theta - \frac{\pi}{4}$ is from 0 to $\frac{1}{k-1}\frac{\pi}{2}$ and the range of $\frac{k-1}{4}\theta + \frac{\pi}{4}$ is from $\frac{\pi}{2} - \frac{1}{k-1}\frac{\pi}{2}$ to $\frac{\pi}{2}$. Consequently (5.14) shows that $\sin \frac{k}{2}\theta \geq \cos \frac{1}{2}\theta$ as required. When $k = 1$ and θ is in the interval $[\frac{\pi}{2}, \pi]$ again formula (5.14) establishes the result.

To conclude the argument, $|\bar{g}(\theta)| = |\sin \frac{N}{2}\theta \cos \frac{1}{2}\theta| \leq \cos \frac{1}{2}\theta$, and when N is odd the collection of \bar{f}_r 's is $\sin \theta, \sin 2\theta, \dots, \sin \frac{1}{2}(N-1)\theta$, so we may exclude

$$[\frac{\pi}{N}, \pi] = [\frac{\pi}{3}, \pi] \cup [\frac{\pi}{5}, \frac{\pi}{3}] \cup [\frac{\pi}{7}, \frac{\pi}{5}] \cup \dots \cup [\frac{\pi}{N}, \frac{\pi}{N-2}],$$

but when N is even the collection of \bar{f}_r 's is $\sin \frac{1}{2}\theta, \sin \frac{3}{2}\theta, \dots, \sin \frac{1}{2}(N-1)\theta$ and we may exclude

$$[\frac{\pi}{N}, \pi] = [\frac{\pi}{2}, \pi] \cup [\frac{\pi}{4}, \frac{\pi}{2}] \cup [\frac{\pi}{6}, \frac{\pi}{4}] \cup \dots \cup [\frac{\pi}{N}, \frac{\pi}{N-2}].$$

□

Further strings generating periodic sequences, with intervals of admissibility near $\theta = \pi$, may be derived from the strings $+-0\dots 0$ that we have been considering. To do this we again exploit symmetry by means of Proposition 2.4. In the present case the single family of sequences generated by strings $+-0\dots 0$ admissible near $\theta = 0$ gives rise to two distinct families of sequences near $\theta = \pi$.

- Sequences generated by strings $+-0\dots 0$ of even length N give rise to sequences generated by strings $++0\dots 0$ of length N with interval of admissibility $(\frac{(N-1)\pi}{N}, \pi)$.
- Sequences generated by strings $+-0\dots 0$ of odd length N give rise to sequences generated by strings $++0\dots 0--0\dots 0$ of length $2N$ with interval of admissibility $(\frac{(N-1)\pi}{N}, \pi)$.

The first few members of these infinite families are listed in Table 5.6, and further examples of sequences from these families are to be found in Table 1 of Volume II, in the lists of admissible periodic sequences for values of θ near π .

| String Length (N) | Generating String | Interval of Admissibility |
|--------------------------|-------------------|-----------------------------|
| 4 | ++00 | $(\frac{3\pi}{4}, \pi)$ |
| 6 | ++0000 | $(\frac{5\pi}{6}, \pi)$ |
| 8 | ++000000 | $(\frac{7\pi}{8}, \pi)$ |
| 10 | ++00000000 | $(\frac{9\pi}{10}, \pi)$ |
| \vdots | \vdots | \vdots |
| N | ++0...0 | $(\frac{(N-1)\pi}{N}, \pi)$ |

| String Length (N) | Generating String | Interval of Admissibility |
|--------------------------|---------------------------|-----------------------------|
| 6 | ++0--0 | $(\frac{2\pi}{3}, \pi)$ |
| 10 | ++000--000 | $(\frac{4\pi}{5}, \pi)$ |
| 14 | ++00000--00000 | $(\frac{6\pi}{7}, \pi)$ |
| 18 | ++0000000--0000000 | $(\frac{8\pi}{9}, \pi)$ |
| \vdots | \vdots | \vdots |
| N | ++0...0--0...0 | $(\frac{(N-2)\pi}{N}, \pi)$ |

Table 5.6: The first few generating strings for the two families derived from strings **++0...0** by the symmetry result.

Note that *all* admissible sequences generated by strings **++0...0** actually arise in this way (via the symmetry) from sequences generated by the strings **+-0...0**. This is because no odd length string **++0...0** can generate an admissible periodic sequence, another consequence of the result from Chapter 3 that a string **++0...0** of odd length may not appear in *any* admissible sequence.

The family of sequences generated by the strings **+-0...0** are by no means the only ones to have left end-point of admissibility at $\theta = 0$, and this represents an inherent dif-

ference between the behaviour of the system near $\theta = \frac{\pi}{2}$ and the behaviour near $\theta = 0$. Reference to the lists of sequences in Table 1 of Volume II (for example that for $\theta = 0.15$) shows that the computer searches have turned up very many sequences which would appear to have $\theta = 0$ as the left end-point of an interval of admissibility. Investigating precisely which sequences can arise is the subject of Chapters 7 and 8.

5.3 Periodic sequences admissible in an interval having $\frac{\pi}{3}$ as an end-point

Up to this point in the chapter we have demonstrated the existence of several infinite families of admissible periodic sequences, whose intervals of admissibility have an end-point at $\theta = 0, \frac{\pi}{2}$ or π . These locations for the parameter θ coincide with regions of clustering of those admissible periodic sequences that we found with our computer searches : clustering shows up as broad peaks in the graph of Figure 4.8 in Chapter 4, where we plotted the number of periodic sequences admissible for the parameter value θ against θ . Elsewhere on this graph there is a noticeable peak for parameter values around $\frac{\pi}{3}$ (and also around $\frac{2\pi}{3}$, but the admissible periodic sequences arising here may be related to those at $\frac{\pi}{3}$ via the transformation that negates alternate digits of the generating string). Searches conducted immediately to either side of $\theta = \frac{\pi}{3}$, as reported in Table 4 of Volume II, show clear evidence for the existence of further infinite families of admissible periodic sequences. We isolated members of two such families in the presentation of our computational results in Section 4.2.4, each consisting of a run of alternating + and - digits followed by a run of 0's. More definitely, we propose here the following four infinite families, each parametrised by the integer $m = 1, 2, \dots$: two families having $\frac{\pi}{3}$ as the *left* end-point of an interval of admissibility

(a) $(+-)^{3m-1}0^{6m-1}$, with length $N = 12m - 3$ and first member $+-+00000$,

(b) $(+-)^{3m}0^{6m+2}$, with length $N = 12m + 3$ and first member $+-+-+00000000$,

and two families having $\frac{2\pi}{3}$ as the *right* end-point of an interval of admissibility

- (c) $(+-)^{3m-1}0^{6m-2}$, with length $N = 12m - 3$ and first member $+-+-+0000$,
- (d) $(+-)^{3m+1}0^{6m+1}$, with length $N = 12m + 3$ and first member $+-+-+-+0000000$.

In this section we undertake a detailed investigation for the family (a), and verify that these strings do generate admissible periodic sequences as conjectured. Demonstrating that the Chua-Lin inequalities hold for some interval extending to the right of $\frac{\pi}{3}$ and obtaining an estimate for the size of that interval in terms of the string length proves to be a much more challenging task than it was for the families of sequences studied in Sections 5.1 and 5.2. We envisage that the techniques we introduce to treat family (a) can be adapted in minor ways to be effective also for the families (b), (c) and (d).

5.3.1 The functions which appear in the Chua-Lin inequalities

For the family of periodic sequences under investigation, each function f_r , the subject of one of the Chua-Lin inequalities, is an alternating sum of cosine terms. Left in this form such functions are difficult to analyse, a multiplicative expression is more useful for comparison with $g(\theta) = \sin \frac{N}{2}\theta \sin \theta$. Trigonometric sums of the type met here are closely related to geometric progressions, and can be recast in compact form. The following lemma shows how this may be achieved and will enable us to write down more convenient expressions for the f_r 's.

Lemma 5.7

$$\cos a\theta - \cos(a-1)\theta + \cos(a-2)\theta - \dots \pm \cos b\theta = \frac{\cos(a + \frac{1}{2})\theta \pm \cos(b - \frac{1}{2})\theta}{2 \cos \frac{1}{2}\theta}.$$

(The sign of the last term depends on whether an odd or an even number of terms are included in the sum.)

Proof

Let

$$z = e^{ai\theta} - e^{(a-1)i\theta} + e^{(a-2)i\theta} - \dots \pm e^{bi\theta},$$

then

$$e^{-i\theta}z = e^{(a-1)i\theta} - e^{(a-2)i\theta} + \dots \mp e^{bi\theta} \pm e^{(b-1)i\theta},$$

and adding these two gives

$$(1 + e^{-i\theta})z = e^{ai\theta} \pm e^{(b-1)i\theta}.$$

Thus

$$(e^{\frac{1}{2}i\theta} + e^{-\frac{1}{2}i\theta})z = e^{(a+\frac{1}{2})i\theta} \pm e^{(b-\frac{1}{2})i\theta},$$

and taking real parts we have

$$2 \cos \frac{1}{2}\theta (\operatorname{Re} z) = \cos(a + \frac{1}{2})\theta \pm \cos(b - \frac{1}{2})\theta,$$

but $\operatorname{Re} z = \cos a\theta - \cos(a-1)\theta + \cos(a-2)\theta - \dots \pm \cos b\theta$, the series to be summed. \square

Unlike the families of sequences that have been investigated in the earlier sections of this chapter, it is not possible to specify all the f_r 's collectively in a single formula parametrised by r . Two structurally different types of function arise for the present family and these need to be treated separately. To make clear the nature of this difference, we consider first the task of writing down the f_r 's in a particular instance, namely when $N = 21$. The generating string is $+-+--+-+--00000000000$, which consists of 5 $+-$ pairs followed by 11 zeros. The 21 functions f_r arising in the corresponding Chua-Lin inequalities are given by the formula

$$f_r = \sum_{i=0}^{20} s_{r+i} \cos \frac{21-2i}{2}\theta \quad (5.15)$$

(where, as before, values of the coefficients s_{r+i} beyond 20 are generated by periodicity). We can view this formula as producing the functions f_r by taking in turn the 21 variants of the string obtained by cycling, and using each to provide coefficients for the cosine

functions in the sum. They are all listed in Table 5.7, where each column is headed by the multiplier of θ in the argument of the appropriate cosine function; the 21 functions can be immediately read from the rows of the table using formula (5.15).

| | | Multiplier of θ in the argument of the cosine term of equation (5.15) | | | | | | | | | | | | | | | | | | | | |
|----------|-------------|--|----------------|----------------|----------------|----------------|----------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|---------------|----------------|----------------|----------------|----------------|----------------|
| Function | | $\frac{21}{2}$ | $\frac{19}{2}$ | $\frac{17}{2}$ | $\frac{15}{2}$ | $\frac{13}{2}$ | $\frac{11}{2}$ | $\frac{9}{2}$ | $\frac{7}{2}$ | $\frac{5}{2}$ | $\frac{3}{2}$ | $\frac{1}{2}$ | $\frac{1}{2}$ | $\frac{3}{2}$ | $\frac{5}{2}$ | $\frac{7}{2}$ | $\frac{9}{2}$ | $\frac{11}{2}$ | $\frac{13}{2}$ | $\frac{15}{2}$ | $\frac{17}{2}$ | $\frac{19}{2}$ |
| f_0 | ϕ_0 | + | - | + | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| f_1 | $-\psi_1$ | - | + | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + |
| f_2 | ψ_4 | + | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - |
| f_3 | $-\psi_2$ | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + |
| f_4 | ψ_3 | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - |
| f_5 | $-\psi_3$ | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + |
| f_6 | ψ_2 | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - |
| f_7 | $-\psi_4$ | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - |
| f_8 | ψ_1 | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - | + | - |
| f_9 | ϕ_{12} | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - | + | - | + |
| f_{10} | ϕ_{11} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - | + | - | + | - |
| f_{11} | ϕ_{10} | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - | + | - | + | - | 0 |
| f_{12} | ϕ_9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - | + | - | + | - | 0 | 0 |
| f_{13} | ϕ_8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - | + | - | + | - | 0 | 0 | 0 |
| f_{14} | ϕ_7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 |
| f_{15} | ϕ_6 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 |
| f_{16} | ϕ_5 | 0 | 0 | 0 | 0 | 0 | + | - | + | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 |
| f_{17} | ϕ_4 | 0 | 0 | 0 | 0 | + | - | + | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| f_{18} | ϕ_3 | 0 | 0 | 0 | + | - | + | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| f_{19} | ϕ_2 | 0 | 0 | + | - | + | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| f_{20} | ϕ_1 | 0 | + | - | + | - | + | - | + | - | + | - | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 5.7: The 21 variants of the string $+-+--+-+--0000000000$ obtained by cycling. To the left of each variant is the function deriving from that particular string.

Many of the rows correspond to an f_r which is a regularly progressing alternating sum of cosine terms; to be precise the functions are $f_0, f_{20}, f_{19}, f_{18}, \dots, f_{10}, f_9$. In the other rows the run of $+-$ pairs wraps around from the end of the row back to the beginning. Consequently the sum (5.15) includes $\pm (\cos \frac{19}{2}\theta - \cos \frac{21}{2}\theta + \cos \frac{19}{2}\theta)$, which it is impossible to incorporate into a single trigonometric sum of the sort appearing in Lemma 5.7. The best

that can be achieved in these cases is to treat the function f_r as the sum of two individual alternating sums, taken either side of the wrap-around.

It will be convenient in what follows to introduce separate notation for the two types of function f_r described above; the ϕ_r 's will be those expressible by a single regular alternating sum, and the ψ_r 's those requiring two such sums because of wrap-around. Exactly how the ϕ_r 's and ψ_r 's correlate with the original f_r 's in general we will describe shortly. The correspondence between the ϕ_r 's and ψ_r 's and the f_r 's for the case $N = 21$ is recorded in the second column of Table 5.7.

Summing the relevant trigonometric series by means of Lemma 5.7 we obtain the following expressions for the functions that we need to consider :

$$\begin{aligned}\phi_r &= \frac{\cos(11-r)\theta - \cos(r-1)\theta}{2 \cos \frac{1}{2}\theta} & \text{for } r = 0, 1, \dots, 12, \\ \psi_r &= \frac{\cos 11\theta - \cos(11-2r)\theta + \cos 2r\theta - \cos 10\theta}{2 \cos \frac{1}{2}\theta} & \text{for } r = 1, 2, 3, 4,\end{aligned}$$

and

$$\psi_{-r} = -\psi_r \quad \text{for } r = 1, 2, 3, 4.$$

Note that the negated versions of the ϕ_r 's, as well as the ψ_r 's, are present, since $\phi_{12-r} = -\phi_r$.

In the general case, where $N = 12m - 3$, the formulae for the ϕ_r 's and ψ_r 's are :

$$\begin{aligned}\phi_r &= \frac{\cos(6m-1-r)\theta - \cos(r-1)\theta}{2 \cos \frac{1}{2}\theta} & [\text{additive form}] \\ &= -\frac{\sin(3m-1)\theta \sin(3m-r)\theta}{\cos \frac{1}{2}\theta} & [\text{multiplicative form}]\end{aligned} \tag{5.16}$$

for $r = 0, 1, \dots, 6m$,

$$\psi_r = \frac{\cos(6m-1)\theta - \cos((6m-1)-2r)\theta + \cos 2r\theta - \cos(6m-2)\theta}{2 \cos \frac{1}{2}\theta} \tag{5.17}$$

for $r = 1, 2, \dots, 3m-2$,

$$\psi_{-r} = -\psi_r \quad \text{for } r = 1, 2, \dots, 3m-2. \tag{5.18}$$

These formulae give a sufficiently simple collective description of the f_r 's to be effective in the subsequent work on the family of sequences under investigation. Both additive and multiplicative forms for the ϕ_r 's are included because both are useful at different stages of the calculations that come later. As a check, note that the total number of functions listed above is $(6m+1) + (3m-2) + (3m-2) = 12m-3 = N$. As pointed out for the case $N = 21$, the collection of ϕ_r 's includes negations, since $\phi_{6m-r} = -\phi_r$.

It is important to realise that the ϕ_r 's and ψ_r 's have been introduced simply to take into account the two types of function present amongst the f_r 's, and that each is one of the functions f_r appearing in the Chua-Lin inequalities. The actual correspondence between the ϕ_r 's, ψ_r 's and f_r 's in the general case is as follows :

$$\begin{aligned}\phi_0 &= f_0, \\ \phi_r &= f_{N-r} \quad \text{for } r = 1, 2, \dots, 6m,\end{aligned}\tag{5.19}$$

$$\left. \begin{aligned}\psi_r &= f_{\frac{1}{2}(N-1)-2r} \\ \psi_{-r} &= f_{2r-1}\end{aligned} \right\} \quad \text{for } r = 1, 2, \dots, 3m-2.\tag{5.20}$$

5.3.2 Existence of an interval of admissibility

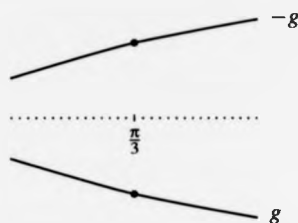
Our first step in the analysis of the family of sequences we are currently considering is to show that there actually is an interval of admissibility with left end-point $\theta = \frac{\pi}{3}$ for each member of the family. Later we will estimate the length of the interval. The procedure we adopt here to establish the existence of an interval of admissibility is similar to that of Section 5.2, where we showed the maximality of the intervals of admissibility for the sequences considered there. We use derivatives to show that all the functions ϕ_r and ψ_r , i.e. the f_r 's, enter the envelope of $\pm g$ at $\theta = \frac{\pi}{3}$.

Now

$$g(\theta) = \sin \frac{N}{2}\theta \sin \theta = \sin \frac{12m-3}{2}\theta \sin \theta,\tag{5.21}$$

$$\begin{aligned}g'(\theta) &= \frac{N}{2} \cos \frac{N}{2}\theta \sin \theta + \sin \frac{N}{2}\theta \cos \theta \\ &= \frac{12m-3}{2} \cos \frac{12m-3}{2}\theta \sin \theta + \sin \frac{12m-3}{2}\theta \cos \theta,\end{aligned}\tag{5.22}$$

and setting $\theta = \frac{\pi}{3}$ we find that $g(\frac{\pi}{3}) = -\frac{\sqrt{3}}{2}$ and $g'(\frac{\pi}{3}) = -\frac{1}{2}$. Close to $\theta = \frac{\pi}{3}$, the graphs of g and $-g$ are depicted below.



The corresponding values of the functions ϕ_r and their first derivatives at $\theta = \frac{\pi}{3}$ are calculated from the formulae (5.16), and are listed in the following table :

| r congruent (mod 6) to | $\phi_r(\frac{\pi}{3})$ | $\phi'_r(\frac{\pi}{3})$ |
|-----------------------------|-------------------------|--------------------------|
| 0 | 0 | not needed |
| 1 | $-\frac{\sqrt{3}}{2}$ | $\frac{1}{4}(12m-3-2r)$ |
| 2 | $-\frac{\sqrt{3}}{2}$ | $\frac{1}{4}(2r-3)$ |
| 3 | 0 | not needed |
| 4 | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{4}(12m-3-2r)$ |
| 5 | $\frac{\sqrt{3}}{2}$ | $-\frac{1}{4}(2r-3)$ |

When r is congruent (mod 6) to 0 or 3 the functions ϕ_r take the value 0 at $\theta = \frac{\pi}{3}$, which is well between the values of g and $-g$ there, so these ϕ_r certainly lie within the envelope of $\pm g$ for a short way to either side of $\theta = \frac{\pi}{3}$. All other functions ϕ_r intersect either g or $-g$ at $\theta = \frac{\pi}{3}$. Consequently we turn to the information about the derivatives in order to determine whether they are inside or outside the envelope of $\pm g$ to either side of $\theta = \frac{\pi}{3}$. That any remaining ϕ_r enters the envelope of $\pm g$ as θ increases from $\frac{\pi}{3}$, which we claim to be the case, will follow if we show that $\phi_r(\frac{\pi}{3}) \phi'_r(\frac{\pi}{3}) < 0$. When $r \equiv 2$ or $5 \pmod{6}$ this requires $2r-3 > 0$, which is clearly true; when $r \equiv 1$ or $4 \pmod{6}$ we require $12m-3-2r > 0$, which is so because here $r \leq 6m-2$.

We treat the functions ψ_r similarly, in order to check that they too enter the envelope of $\pm g$ at $\frac{\pi}{3}$. The values of the ψ_r 's and their first derivatives at $\theta = \frac{\pi}{3}$ are as follows :

| r congruent (mod 3) to | $\psi_r\left(\frac{\pi}{3}\right)$ | $\psi'_r\left(\frac{\pi}{3}\right)$ |
|-----------------------------|------------------------------------|-------------------------------------|
| 0 | $\frac{\sqrt{3}}{2}$ | $\frac{5}{4} + r - 3m$ |
| 1 | $\frac{\sqrt{3}}{2}$ | $\frac{3}{4} - r$ |
| 2 | 0 | not needed |

To check the condition $\psi_r\left(\frac{\pi}{3}\right)\psi'_r\left(\frac{\pi}{3}\right) < 0$ it suffices to recall that in the case of the ψ_r 's the range for r is $1, 2, \dots, 3m-2$, so that $r \leq 3m-3$ when $r \equiv 0 \pmod{3}$. Of course when each ψ_r is within the envelope of $\pm g$, so is each of the functions $-\psi_r$.

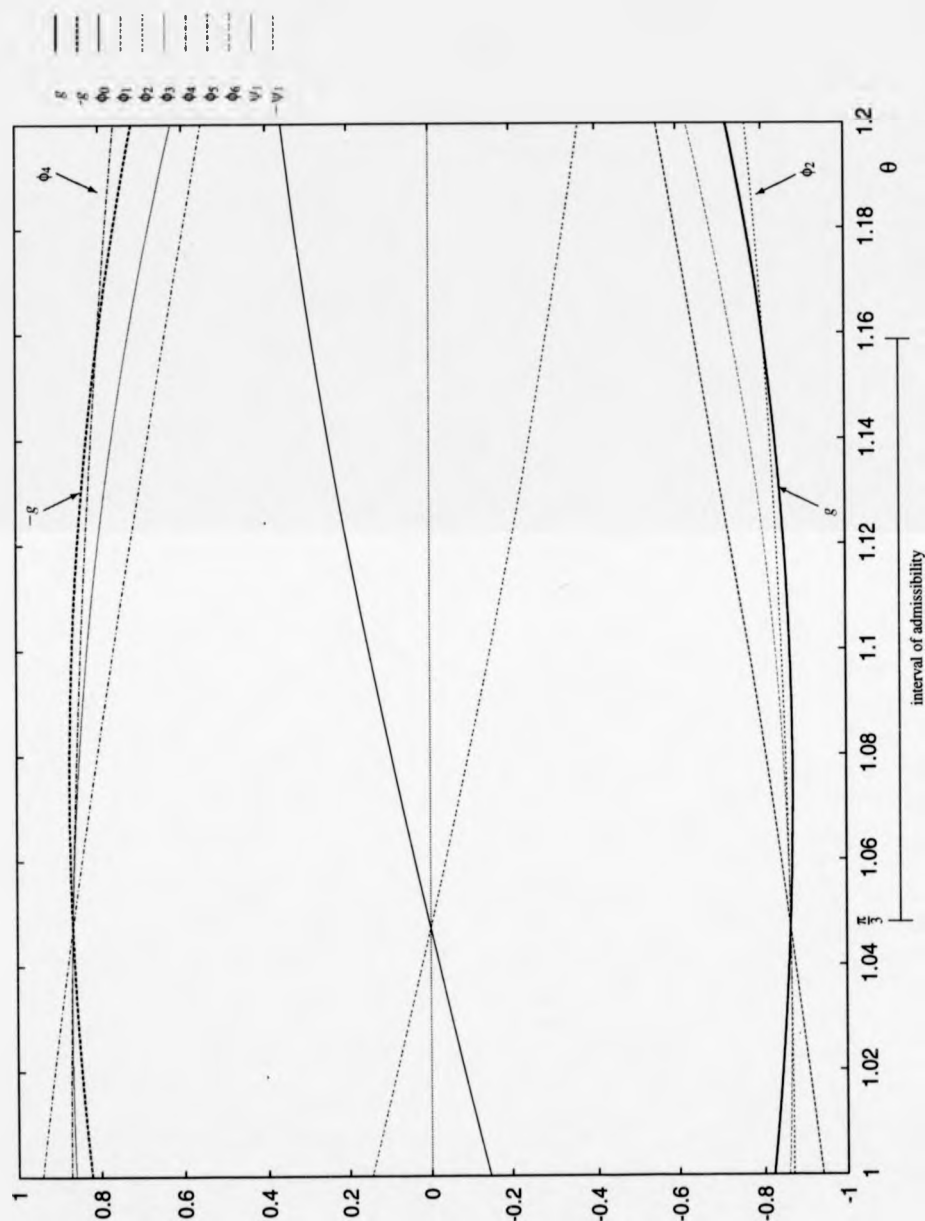
The consequence of the foregoing is that, by continuity, for some (short) interval starting at $\theta = \frac{\pi}{3}$ and extending to the right all the f_r 's lie within the envelope of $\pm g$, and hence the periodic sequence is admissible throughout that interval. Immediately to the left of $\theta = \frac{\pi}{3}$, some of the f_r 's leave the envelope of g and $-g$ (the derivatives show this), so the periodic sequence fails to be admissible.

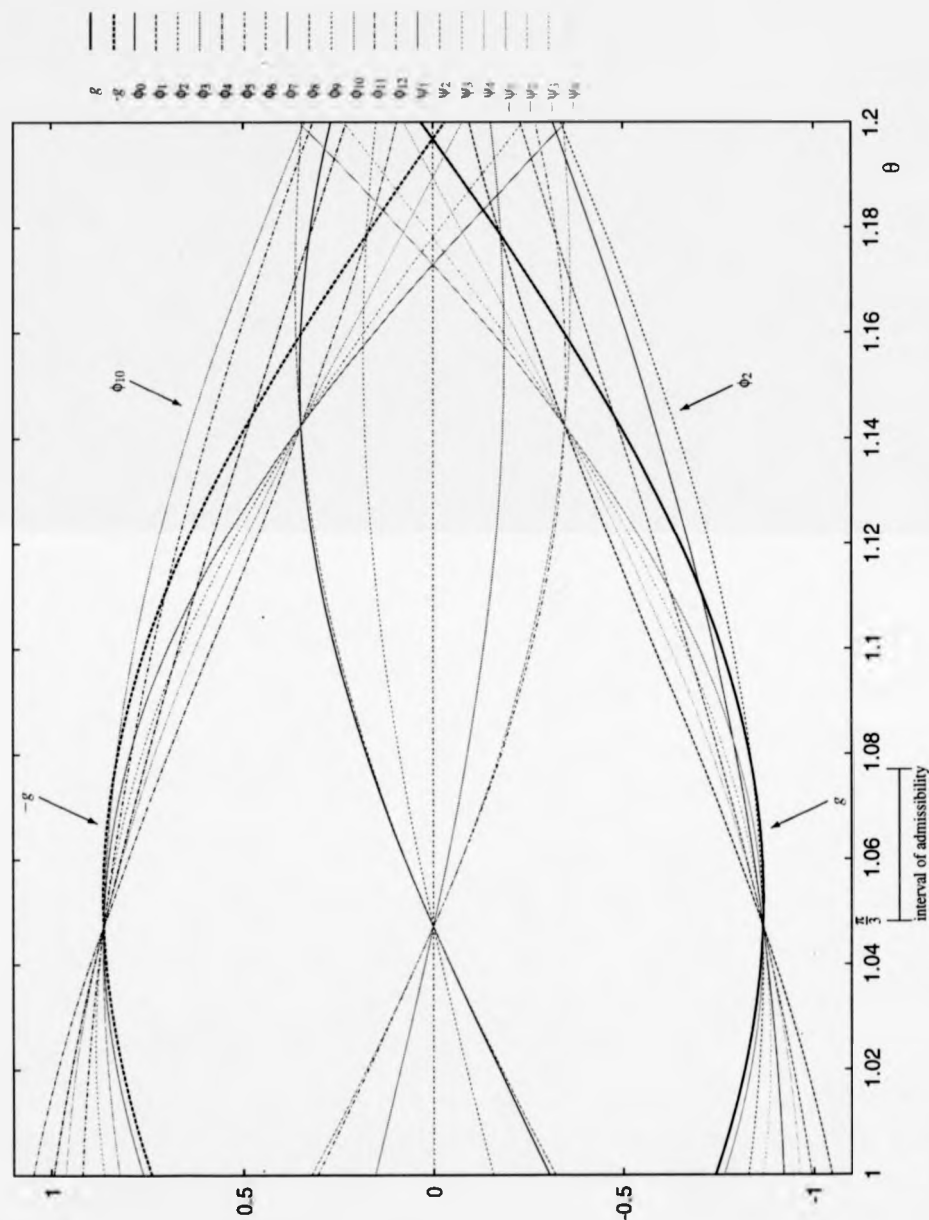
5.3.3 A dominating function

The key step in demonstrating the admissibility of the periodic sequences considered in Sections 5.1 and 5.2 was to show that over the appropriate interval for θ there was a single f_r which dominated the rest, so that essentially this function determined the interval of admissibility (via the associated one of the Chua-Lin inequalities). In this section we show that the same technique can be applied in the present situation, but there are various complications to be overcome which were not previously encountered.

Figures 5.6 and 5.7 are plots of the functions f_r and g in the vicinity of $\theta = \frac{\pi}{3}$ for the first two generating strings (which correspond to $m = 1$ and $m = 2$), in each case showing the interval to the right of $\frac{\pi}{3}$ for which all the f_r lie within the envelope of $\pm g$. In both there is a single function f_r (not counting its negation, $-f_r$) which dominates throughout the interval: the function is ϕ_{6m-2} in both instances.

It is thus our aim to show in general that the function ϕ_{6m-2} dominates, in absolute value, all the ϕ_r 's and the ψ_r 's over the interval of admissibility. Unfortunately, since

Figure 5.6: Plot of the functions ϕ_r , ψ , and $\pm g$ for the string $+-+--00000$ of length $N=9$ in the vicinity of $\theta = \frac{\pi}{3}$.

Figure 5.7: The corresponding plot for the next string belonging to the family, $+-+--+-+--000000000000$ of length $N = 21$.

the right end-point of the interval does not appear to be a simple multiple of π for the family of sequences under consideration, we cannot, as was possible previously, make a conjecture for the extent of the interval of admissibility, and subsequently verify that the selected function dominates throughout. Instead we specify a larger interval over which ϕ_{6m-2} dominates, and later check that this larger interval in fact does include the interval of admissibility.

In Figure 5.7, we can see that there is a point just to the right of $\theta = 1.14$ where several of the graphs of the functions intersect. To be precise, this point is located at $\theta = \frac{4\pi}{11}$ and is the first θ value to the right of $\frac{\pi}{3}$ where the graphs of ψ_1, ψ_2, ψ_3 and ψ_4 all intersect. Here we have been considering $N = 21$; for the next case, $N = 33$, there is a similar point where all of ψ_1 to ψ_7 intersect, and so on. This suggests a means for determining a simple rational multiple of π , which can be taken as the right end-point of an interval over which we intend to show that ϕ_{6m-2} dominates.

For the moment, we will obtain the location of the point in general by seeking where the graphs of ψ_1 and ψ_2 first meet to the right of $\frac{\pi}{3}$; later it will become apparent that all the ψ_r 's meet at this point. From the formulae (5.17), ψ_1 and ψ_2 take a common value when

$$\begin{aligned} \cos(6m-1)\theta - \cos((6m-1)-2)\theta + \cos 2\theta - \cos(6m-2)\theta \\ = \cos(6m-1)\theta - \cos((6m-1)-4)\theta + \cos 4\theta - \cos(6m-2)\theta. \end{aligned} \quad (5.23)$$

After several simplifying steps (transforming trigonometric sums to products and jettisoning a non-vanishing $\sin \theta$ factor) this equation reduces to

$$\sin \frac{1}{2}(6m-1)\theta \cos \frac{1}{2}(6m+7)\theta = 0. \quad (5.24)$$

The first solution to this equation for θ beyond $\frac{\pi}{3}$ is $\theta = \frac{2m\pi}{6m-1}$; for notational convenience we set $\eta = \frac{2m\pi}{6m-1}$. (In the exceptional case $m = 1$ we still use this formula to define η , even though there is just the single function ψ_1 present.) The rest of this section is concerned with showing that ϕ_{6m-2} dominates all the other ϕ_r 's and all the ψ_r 's over the interval $(\frac{\pi}{3}, \eta)$.

To show that ϕ_{6m-2} does dominate the other functions proves to be a lengthy task. The strategy we adopt involves showing the following for $\frac{\pi}{3} \leq \theta \leq \eta$:

1. $\phi_{6m-2} > 0$, and $\phi_{6m-2} \geq |\phi_r|$ for all r ,
2. $\psi_r \geq 0$, and $\psi_1 \geq \psi_r$ for all r ,
3. $\phi_{6m-2} \geq \psi_1$.

We consider each of these steps in turn.

Step 1: ϕ_{6m-2} dominates all other ϕ_r 's

The ϕ_r 's are the easier set of functions to treat. This is because the multiplicative expression for a function ϕ_r resembles that for each of the functions f_r considered in Sections 5.1 and 5.2; a similar style of argument involving the ratio of two sine functions establishes the claim. Starting with the multiplicative expression for ϕ_r from (5.16)

$$\phi_r(\theta) = \frac{-\sin(3m-1)\theta \sin(3m-r)\theta}{\cos \frac{1}{2}\theta},$$

and so

$$\frac{\phi_r(\theta)}{\phi_{6m-2}(\theta)} = -\frac{\sin(3m-r)\theta}{\sin(3m-2)\theta}.$$

To analyse the variation of this ratio, and also for subsequent tasks, it will be more convenient to effect a change of origin by expressing values of θ in the interval $[\frac{\pi}{3}, \eta]$ as $\theta = \frac{\pi}{3} + \alpha$, so that the variable α assumes (small) values in the range from 0 to $\delta = \eta - \frac{\pi}{3} = \frac{\pi}{3(6m-1)}$. Replacing θ by $\frac{\pi}{3} + \alpha$, and supposing $0 \leq r \leq 3m$ (this is no restriction because $\phi_{6m-r} = -\phi_r$), gives

$$\frac{|\phi_r(\theta)|}{|\phi_{6m-2}(\theta)|} = \begin{cases} \frac{\sin(3m-r)\alpha}{\sin(\frac{\pi}{3} + (3m-2)\alpha)} & \text{when } r \equiv 0 \pmod{3} \\ \frac{\sin(\frac{\pi}{3} - (3m-r)\alpha)}{\sin(\frac{\pi}{3} + (3m-2)\alpha)} & \text{when } r \equiv 1 \pmod{3} \\ \frac{\sin(\frac{\pi}{3} + (3m-r)\alpha)}{\sin(\frac{\pi}{3} + (3m-2)\alpha)} & \text{when } r \equiv 2 \pmod{3}. \end{cases} \quad (5.25)$$

The argument $\frac{\pi}{3} + (3m-2)\alpha$ remains in the first quadrant as α increases from 0 to $\delta = \frac{\pi}{3(6m-1)}$. When $0 \leq r \leq 3m$, the sine function of the numerator does not exceed the sine function in the denominator.

The final detail here is to show that $\phi_{6m-2}(\theta) > 0$ when $\frac{\pi}{3} \leq \theta \leq \eta$. Now

$$\phi_{6m-2}(\theta) = \frac{\sin(3m-1)\theta \sin(3m-2)\theta}{\cos \frac{1}{2}\theta} \quad (5.26)$$

and it is sufficient to verify that the numerator is positive. Taking, as before, $\theta = \frac{\pi}{3} + \alpha$,

$$\sin(3m-1)\theta \sin(3m-2)\theta = \sin\left(\frac{\pi}{3} - (3m-1)\alpha\right) \sin\left(\frac{\pi}{3} + (3m-2)\alpha\right) \quad (5.27)$$

which is strictly positive when $0 \leq \alpha \leq \frac{\pi}{3(6m-1)}$.

Step 2: ψ_1 dominates all other ψ_r 's

First we will show that each $\psi_r \geq 0$ for $\frac{\pi}{3} \leq \theta \leq \eta$. This is certainly evident in the examples plotted in Figures 5.6 and 5.7, but the general argument is rather intricate. Recall that

$$\psi_r(\theta) = \frac{1}{2 \cos \frac{1}{2}\theta} \{ \cos(6m-1)\theta - \cos(6m-1-2r)\theta + \cos 2r\theta - \cos(6m-2)\theta \}$$

and that the values of the ψ_r 's at $\theta = \frac{\pi}{3}$ are :

| r congruent (mod 3) to | $\psi_r\left(\frac{\pi}{3}\right)$ |
|-----------------------------|------------------------------------|
| 0 | $\frac{\sqrt{3}}{2}$ |
| 1 | $\frac{\sqrt{3}}{2}$ |
| 2 | 0 |

Expressed in terms of the variable α , where $\theta = \frac{\pi}{3} + \alpha$, we find that ψ_r becomes

$$\begin{aligned} 2 \cos \frac{1}{2}\theta \psi_r(\theta) = & \cos\left(-\frac{\pi}{3} + (6m-1)\alpha\right) - \cos\left(-\frac{(2r+1)\pi}{3} + (6m-1-2r)\alpha\right) \\ & + \cos\left(\frac{2r\pi}{3} + 2r\alpha\right) - \cos\left(-\frac{2\pi}{3} + (6m-2)\alpha\right). \end{aligned} \quad (5.28)$$

Recall that $1 \leq r \leq 3m-2$. We find it necessary to treat separately the three cases $r \equiv 0, 1$ or $2 \pmod{3}$; the fact that $\psi_r(\frac{\pi}{3}) = 0$ when $r \equiv 2 \pmod{3}$ means that this case is essentially different from the other two and it requires a more delicate argument.

Case 1 : $r \equiv 0 \pmod{3}$

$$2 \cos \frac{1}{2} \theta \psi_r(\theta) = \cos\left(-\frac{\pi}{3} + (6m-1)\alpha\right) - \cos\left(-\frac{\pi}{3} + (6m-1-2r)\alpha\right) \\ + \cos 2r\alpha - \cos\left(-\frac{2\pi}{3} + (6m-2)\alpha\right), \quad (5.29)$$

and

$$\begin{aligned} \cos\left(-\frac{\pi}{3} + (6m-1)\alpha\right) - \cos\left(-\frac{\pi}{3} + (6m-1-2r)\alpha\right) \\ = -2 \sin\left(-\frac{\pi}{3} + (6m-1)\alpha\right) \sin r\alpha \\ = 2 \cos\left(\frac{\pi}{6} + (6m-1-r)\alpha\right) \sin r\alpha \\ > 0, \end{aligned}$$

$$\begin{aligned} \cos 2r\alpha - \cos\left(-\frac{2\pi}{3} + (6m-2)\alpha\right) \\ = 2 \cos\left(\frac{\pi}{6} + (3m-1+r)\alpha\right) \cos\left(\frac{\pi}{6} + (3m-1-r)\alpha\right) \\ > 0. \end{aligned}$$

Case 2 : $r \equiv 1 \pmod{3}$

$$2 \cos \frac{1}{2} \theta \psi_r(\theta) = \cos\left(-\frac{\pi}{3} + (6m-1)\alpha\right) - \cos(-\pi + (6m-1-2r)\alpha) \\ + \cos\left(\frac{2\pi}{3} + 2r\alpha\right) - \cos\left(-\frac{2\pi}{3} + (6m-2)\alpha\right), \quad (5.30)$$

and

$$\begin{aligned} \cos\left(-\frac{\pi}{3} + (6m-1)\alpha\right) + \cos\left(\frac{2\pi}{3} + 2r\alpha\right) \\ = 2 \cos\left(\frac{\pi}{6} + (3m-\frac{1}{2}+r)\alpha\right) \sin\left(3m-\frac{1}{2}-r\right)\alpha \\ > 0, \end{aligned}$$

$$\begin{aligned} \cos((6m-1-2r)\alpha) + \cos\left(\frac{\pi}{3} + (6m-2)\alpha\right) \\ = 2 \cos\left(\frac{\pi}{6} + (6m-\frac{3}{2}-r)\alpha\right) \cos\left(\frac{\pi}{6} + (r-\frac{1}{2})\alpha\right) \\ > 0. \end{aligned}$$

Case 3 : $r \equiv 2 \pmod{3}$

$$2 \cos \frac{1}{2} \theta \psi_r(\theta) = \cos \left(-\frac{\pi}{3} + (6m-1)\alpha \right) - \cos \left(-\frac{5\pi}{3} + (6m-1-2r)\alpha \right) \\ + \cos \left(\frac{4\pi}{3} + 2r\alpha \right) - \cos \left(-\frac{2\pi}{3} + (6m-2)\alpha \right). \quad (5.31)$$

Now

$$\cos \left(-\frac{\pi}{3} + (6m-1)\alpha \right) + \cos \left(\frac{\pi}{3} + (6m-2)\alpha \right) = 2 \cos \left(6m - \frac{3}{2} \right) \alpha \cos \left(\frac{\pi}{3} - \frac{1}{2} \alpha \right) \\ \geq \cos \left(6m - \frac{3}{2} \right) \alpha \quad (5.32)$$

because $\cos \left(\frac{\pi}{3} - \frac{1}{2} \alpha \right) \geq \frac{1}{2}$, and

$$\cos \left(\frac{\pi}{3} + (6m-1-2r)\alpha \right) + \cos \left(\frac{\pi}{3} + 2r\alpha \right) = 2 \cos \left(\frac{\pi}{3} + \left(3m - \frac{1}{2} \right) \alpha \right) \cos \left(3m - \frac{1}{2} - 2r \right) \alpha \\ \leq 2 \cos \left(\frac{\pi}{3} + \left(3m - \frac{1}{2} \right) \alpha \right) \quad (5.33)$$

because $\cos \left(3m - \frac{1}{2} - 2r \right) \alpha \leq 1$; so

$$2 \cos \frac{1}{2} \theta \psi_r(\theta) \geq \cos \left(6m - \frac{3}{2} \right) \alpha - 2 \cos \left(\frac{\pi}{3} + \left(3m - \frac{1}{2} \right) \alpha \right). \quad (5.34)$$

But the function $\cos \left(6m - \frac{3}{2} \right) \alpha - 2 \cos \left(\frac{\pi}{3} + \left(3m - \frac{1}{2} \right) \alpha \right)$ vanishes when $\alpha = 0$ and has derivative

$$(6m-1) \sin \left(\frac{\pi}{3} + \left(3m - \frac{1}{2} \right) \alpha \right) - \left(6m - \frac{3}{2} \right) \sin \left(6m - \frac{3}{2} \right) \alpha \geq \frac{1}{2} \sin \frac{\pi}{3}$$

for $0 \leq \alpha \leq \delta = \frac{\pi}{3(6m-1)}$; it follows that $\psi_r(\theta) \geq 0$.

Now that we know the ψ_r 's to be positive, we move to the task of showing that $\psi_1 \geq \psi_r$ throughout the interval $\left[\frac{\pi}{3}, \eta \right]$. After transforming sums into products we find

$$\cos \frac{1}{2} \theta (\psi_1(\theta) - \psi_r(\theta)) = 2 \sin \left(3m - \frac{1}{2} \right) \theta \sin(r-1)\theta \cos \left(3m - \frac{3}{2} - r \right) \theta. \quad (5.35)$$

The factor $\sin \left(3m - \frac{1}{2} \right) \theta$ has the sign $(-1)^{m+1}$ throughout the interval $\left[\frac{\pi}{3}, \eta \right]$. We replace

θ in each of the other factors by $\frac{\pi}{3} + \alpha$, to obtain

$$\sin(r-1)\theta = \sin\left(\frac{(r-1)\pi}{3} + (r-1)\alpha\right) \quad (5.36)$$

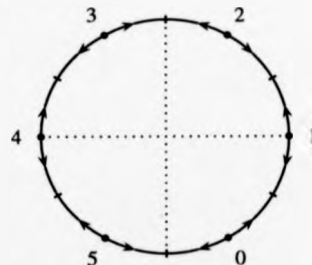
and

$$\begin{aligned} \cos\left(3m - \frac{3}{2} - r\right)\theta &= (-1)^{m+1} \sin\left(\frac{\pi}{3} - \left(3m - \frac{3}{2} - r\right)\alpha\right) \\ &= (-1)^{m+1} \sin\left(\frac{(r-1)\pi}{3} + \left(\frac{\pi}{3} - \left(3m - \frac{3}{2} - r\right)\alpha\right)\right). \end{aligned} \quad (5.37)$$

Since $1 \leq r \leq 3m-2$ and $0 \leq \alpha \leq \delta = \frac{\pi}{3(6m-1)}$, the angles $(r-1)\alpha$ and $\frac{\pi}{3} - (3m - \frac{3}{2} - r)\alpha$ are limited so that

$$0 \leq (r-1)\alpha \leq \frac{\pi}{6} \quad \text{and} \quad \frac{\pi}{6} \leq \frac{\pi}{3} - \left(3m - \frac{3}{2} - r\right)\alpha \leq \frac{\pi}{3}$$

It follows that the factors $\sin\left(\frac{(r-1)\pi}{3} + (r-1)\alpha\right)$ and $\sin\left(\frac{(r-1)\pi}{3} + \left(\frac{\pi}{3} - \left(3m - \frac{3}{2} - r\right)\alpha\right)\right)$ always have the same sign. The quadrant diagram below illustrates this fact; the perimeter integers correlate the residues of r modulo 6 and the angles $\frac{(r-1)\pi}{3}$, anticlockwise arrows represent the argument $\frac{(r-1)\pi}{3} + (r-1)\alpha$, and clockwise arrows represent the argument $\frac{(r-1)\pi}{3} + \left(\frac{\pi}{3} - \left(3m - \frac{3}{2} - r\right)\alpha\right)$.



Note that we have also established our earlier claim that each ψ_r assumes the same value at $\theta = \eta$; this is apparent now because of the vanishing at η of the factor $\sin\frac{1}{2}(6m-1)\theta$ appearing in the difference $\psi_1 - \psi_r$.

Step 3: ϕ_{6m-2} dominates ψ_1

We know that ϕ_{6m-2} dominates all other ϕ_r 's and that ψ_1 dominates all other ψ_r 's. We show

as a final step in this part of the investigation that, throughout the interval $[\frac{\pi}{3}, \eta]$, ϕ_{6m-2} dominates ψ_1 . Because both functions are positive over the interval $[\frac{\pi}{3}, \eta]$, we simply need to show that $\phi_{6m-2} \geq \psi_1$. Since

$$\begin{aligned}
 2 \cos \frac{1}{2}\theta (\phi_{6m-2}(\theta) - \psi_1(\theta)) &= \{\cos \theta - \cos(6m-3)\theta\} - \{\cos(6m-1)\theta \\
 &\quad - \cos(6m-3)\theta + \cos 2\theta - \cos(6m-2)\theta\} \\
 &= \cos \theta - \cos 2\theta + \cos(6m-2)\theta - \cos(6m-1)\theta \\
 &= 2 \sin \frac{1}{2}\theta (\sin \frac{3}{2}\theta + \sin(6m-\frac{3}{2})\theta) \\
 &= 4 \sin \frac{1}{2}\theta \sin 3m\theta \cos(3m-\frac{3}{2})\theta
 \end{aligned} \tag{5.38}$$

and writing $\theta = \frac{\pi}{3} + \alpha$,

$$\sin 3m\theta = \sin(m\pi + 3m\alpha) = (-1)^m \sin 3m\alpha, \tag{5.39}$$

$$\cos(3m-\frac{3}{2})\theta = \cos(m\pi - \frac{\pi}{2} + (3m-\frac{3}{2})\alpha) = (-1)^m \sin(3m-\frac{3}{2})\alpha; \tag{5.40}$$

the product $\sin 3m\theta \cos(3m-\frac{3}{2})\theta \geq 0$ over the interval $[\frac{\pi}{3}, \eta]$, and so $\phi_{6m-2}(\theta) - \psi_1(\theta) \geq 0$.

5.3.4 Estimating the right end-point of the interval of admissibility

Up to this point we have demonstrated that there is, for each periodic sequence of the family under consideration, an interval of admissibility extending to the right from $\frac{\pi}{3}$. We have also found an interval $[\frac{\pi}{3}, \eta]$ and noted one of the f_r 's which dominates all the others throughout this interval. In terms of the alternative notation introduced in Section 5.3.1 the function thus identified is ϕ_{6m-2} . For convenience we shall refer to this function in the remainder of this section simply as ϕ .

In the next lemma we show that the interval of admissibility with left end-point at $\frac{\pi}{3}$ is contained within $[\frac{\pi}{3}, \eta]$. This done we may proceed to locate the right end-point of the interval of admissibility as the point where the function ϕ meets either g or $-g$.

Lemma 5.8

The interval of admissibility is contained within the interval $[\frac{\pi}{3}, \eta]$.

Proof

Recall that $N = 12m - 3$, $g(\theta) = \sin \frac{N}{2}\theta \sin \theta$, $\phi(\theta) = \phi_{6m-2}(\theta) = \frac{\sin(3m-1)\theta \sin(3m-2)\theta}{\cos \frac{1}{2}\theta}$, and $\eta = \frac{2m\pi}{6m-1}$. We already know that ϕ and $-g$ are positive when $\frac{\pi}{3} \leq \theta \leq \eta$, that they assume the same value $\frac{\sqrt{3}}{2}$ when $\theta = \frac{\pi}{3}$, and that $\phi(\theta) < (-g)(\theta)$ throughout some interval to the right of $\frac{\pi}{3}$. To establish the assertion of the lemma it is sufficient to show $\phi(\eta) > (-g)(\eta)$.

Now,

$$(12m - 3) \cdot m\pi = 2m\pi(6m - 1) - m\pi,$$

$$(3m - 1) \cdot 2m\pi = m\pi(6m - 1) - m\pi,$$

$$(3m - 2) \cdot 2m\pi = m\pi(6m - 1) - 3m\pi.$$

So

$$(-g)(\eta) = \sin \frac{1}{2}\eta \sin \eta \tag{5.41}$$

and

$$\begin{aligned} \phi(\eta) &= \frac{(-1)^{m+1} \sin \frac{1}{2}\eta (-1)^{m+1} \sin \frac{3}{2}\eta}{\cos \frac{1}{2}\eta} \\ &= \frac{\sin \frac{1}{2}\eta (\sin \eta \cos \frac{1}{2}\eta + \cos \eta \sin \frac{1}{2}\eta)}{\cos \frac{1}{2}\eta} \\ &= \sin \frac{1}{2}\eta \sin \eta + \sin \frac{1}{2}\eta \tan \frac{1}{2}\eta \cos \eta, \end{aligned} \tag{5.42}$$

and therefore $\phi(\eta) > (-g)(\eta)$. □

The right end-point of the interval of admissibility will be the first point of intersection of the graphs of ϕ and $(-g)$ to the right of $\frac{\pi}{3}$. Equivalently, it will be the first zero of the

function

$$Q(\theta) = 2 \cos \frac{1}{2}\theta ((-g)(\theta) - \phi(\theta)) \quad (5.43)$$

to the right of $\frac{\pi}{3}$ (the introduction of $2 \cos \frac{1}{2}\theta$ simply clears a non-vanishing denominator).

Substituting for g and ϕ gives

$$\begin{aligned} Q(\theta) &= -2 \sin \frac{N}{2}\theta \sin \theta \cos \frac{1}{2}\theta + \cos(6m-3)\theta - \cos \theta \\ &= \frac{1}{2} \{ \cos 6m\theta + \cos(6m-1)\theta - \cos(6m-2)\theta + \cos(6m-3)\theta \} - \cos \theta, \end{aligned} \quad (5.44)$$

and since we are seeking a solution just to the right of $\frac{\pi}{3}$, we change variables via $\theta = \frac{\pi}{3} + \alpha$, to obtain

$$\begin{aligned} Q(\alpha) &= \frac{1}{2} \cos 6m\alpha + \frac{1}{4} \cos(6m-1)\alpha + \frac{\sqrt{3}}{4} \sin(6m-1)\alpha + \frac{1}{4} \cos(6m-2)\alpha \\ &\quad - \frac{\sqrt{3}}{4} \sin(6m-2)\alpha - \frac{1}{2} \cos(6m-3)\alpha - \frac{1}{2} \cos \alpha + \frac{\sqrt{3}}{2} \sin \alpha. \end{aligned} \quad (5.45)$$

This function $Q(\alpha)$ has a zero at $\alpha = 0$ and at least one zero for $0 < \alpha < \delta = \frac{\pi}{3(6m-1)}$.

In the absence of any analytical process of solution for the equation $Q(\alpha) = 0$, we will proceed by finding an approximate solution. Since α is small, we may reasonably employ the approximations $\sin \alpha \approx \alpha$, $\cos \alpha \approx 1 - \frac{1}{2}\alpha^2$. Doing so results in the equation

$$\left(-9m^2 - \frac{9}{2}m + \frac{15}{8}\right)\alpha^2 + \frac{3\sqrt{3}}{4}\alpha = 0, \quad (5.46)$$

which has zeros at $\alpha = 0$ and at

$$\alpha = \frac{2\sqrt{3}}{24m^2 + 12m - 5}. \quad (5.47)$$

This formula, supplying an approximate right end-point for the interval of admissibility, is in excellent agreement with the results from our computational investigations; Table 5.8 compares the two.

| m | String Length (N) | Interval End-point from Computer Search | Estimate for End-point as $\frac{\pi}{3} + \frac{2\sqrt{3}}{24m^2+12m-5}$ |
|-----|--------------------------|--|--|
| 1 | 9 | 1.158222 | 1.158943 |
| 2 | 21 | 1.077167 | 1.077320 |
| 3 | 33 | 1.061177 | 1.061222 |
| 4 | 45 | 1.055293 | 1.055310 |
| 5 | 57 | 1.052478 | 1.052486 |

Table 5.8: Values obtained for the right end-point of the interval of admissibility by the computer search and by the estimate from Section 5.3.4.

5.3.5 Close bounds for the interval of admissibility

Finding an estimate which matches well the data from computational investigations is of considerable interest, but we would like to quantify theoretically the accuracy achieved. Furthermore, we need to be sure that the estimate obtained refers to the right end-point of that interval of admissibility which has left end-point $\frac{\pi}{3}$; up to this point we have not addressed the issue of whether there could be more than one zero of $Q(\alpha)$ within $0 < \alpha < \delta$. If there were more, our estimate could be spurious, and might not describe at all the right end-point of the interval of admissibility.

Here we improve on our estimate by providing upper and lower bounds for the right end-point of the interval of admissibility. Our approach is again to employ polynomial approximations for $Q(\alpha)$, but this time we employ the inequalities

$$\alpha - \frac{1}{6}\alpha^3 \leq \sin \alpha \leq \alpha$$

$$1 - \frac{1}{2}\alpha^2 \leq \cos \alpha \leq 1 - \frac{1}{2}\alpha^2 + \frac{1}{24}\alpha^4$$

which hold for *all* $\alpha \geq 0$, to obtain more precise information. Replacing the sine and cosine terms appearing in $Q(\alpha)$ with the appropriate approximant from these inequalities, we obtain polynomials $P_o(\alpha)$ and $P_{hi}(\alpha)$ for which

$$P_o(\alpha) \leq Q(\alpha) \leq P_{hi}(\alpha)$$

when $\alpha \geq 0$. Explicitly P_{lo} and P_{hi} are :

$$P_{lo}(\alpha) = -\frac{1}{48}((6m-3)^4 + 1)\alpha^4 - \frac{\sqrt{3}}{24}((6m-1)^3 + 2)\alpha^3 - \frac{3}{8}(24m^2 + 12m - 5)\alpha^2 + \frac{3\sqrt{3}}{4}\alpha, \quad (5.48)$$

$$P_{hi}(\alpha) = \frac{1}{96}(2(6m)^4 + (6m-1)^4 + (6m-2)^4)\alpha^4 + \frac{\sqrt{3}}{24}(6m-2)^3\alpha^3 - \frac{3}{8}(24m^2 + 12m - 5)\alpha^2 + \frac{3\sqrt{3}}{4}\alpha. \quad (5.49)$$

For the case $m = 2$ they are plotted in Figure 5.8, together with Q and the diagram confirms that P_{hi} lies above Q , whilst P_{lo} is below Q .

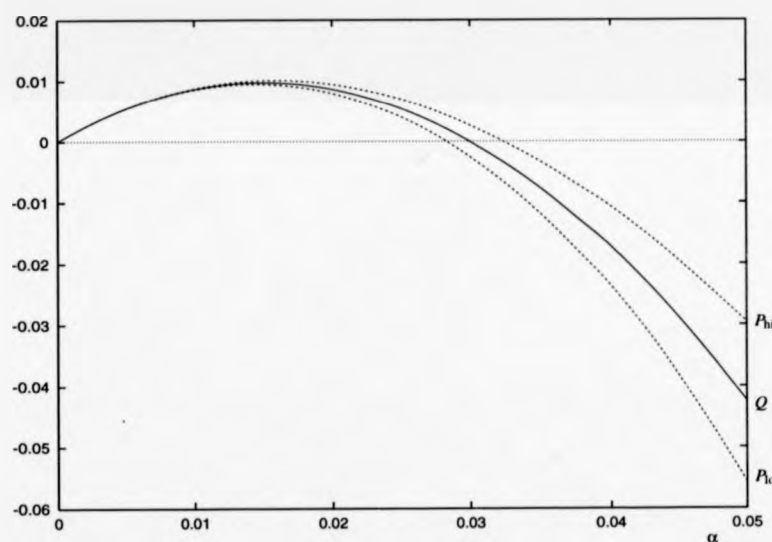


Figure 5.8: Plot of Q and the bounding polynomials P_{lo} and P_{hi} for the case $m = 2$.

Knowing the estimate obtained in Section 5.3.4 for the first positive zero of $Q(\alpha)$, namely

$$\alpha = \frac{2\sqrt{3}}{24m^2 + 12m - 5},$$

we define two approximations

$$\alpha_{lo} = \frac{\sqrt{3}}{12m^2 + 12m}, \quad \alpha_{hi} = \frac{\sqrt{3}}{12m^2}, \quad (5.50)$$

and show that they lie either side of this zero of Q .

Lemma 5.9

The polynomial P_{10} has a single positive zero.

Proof

We remove the factor α from P_{10} to leave the cubic polynomial

$$-\frac{1}{48}((6m-3)^4 + 1)\alpha^3 - \frac{\sqrt{3}}{24}((6m-1)^3 + 2)\alpha^2 - \frac{3}{8}(24m^2 + 12m - 5)\alpha + \frac{3\sqrt{3}}{4}.$$

Differentiating to find the turning points gives

$$-\frac{1}{16}((6m-3)^4 + 1)\alpha^2 - \frac{\sqrt{3}}{12}((6m-1)^3 + 2)\alpha - \frac{3}{8}(24m^2 + 12m - 5) = 0.$$

The discriminant of this quadratic equation is

$$\frac{3}{12^2}((6m-1)^3 + 2)^2 - 4 \cdot \frac{1}{16} \cdot \frac{3}{8}((6m-3)^4 + 1)(24m^2 + 12m - 5),$$

i.e.

$$-1944m^6 + 3402m^5 - \frac{891}{2}m^4 - 2016m^3 + 1458m^2 - \frac{1581}{4}m + \frac{923}{24},$$

and this is negative when $m \geq 2$. Thus for $m \geq 2$ the cubic polynomial has a single zero, necessarily positive.

For the remaining possibility, $m = 1$, the cubic polynomial is

$$-\frac{41}{24}\alpha^3 - \frac{127\sqrt{3}}{24}\alpha^2 - \frac{93}{8}\alpha + \frac{3\sqrt{3}}{4}$$

and its value when $\alpha = -3$ is $81 - \frac{375\sqrt{3}}{8} < 0$, so there are two negative roots and one positive. \square

Now,

$$P_{lo}(\alpha_{lo}) = \frac{2592m^5 + 10584m^4 + 11880m^3 + 960m^2 + 312m - 41}{55296m^4(m+1)^4} \quad (5.51)$$

which is positive for all $m \geq 1$. Thus α_{lo} lies to the left of the positive zero of P_{lo} , and hence to the left of the first zero of Q . Similarly,

$$P_{hi}(\alpha_{hi}) = \frac{-10368m^5 + 3456m^4 + 864m^3 + 696m^2 - 216m + 17}{221184m^8} \quad (5.52)$$

which is negative for all $m \geq 1$, and hence the first zero of Q is to the left of α_{hi} .

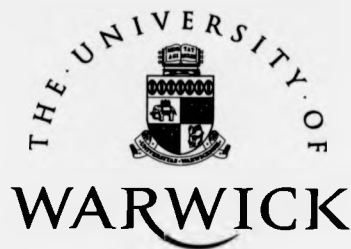
We have thus isolated the right end-point of the interval of admissibility between α_{lo} and α_{hi} , i.e. between $\frac{\sqrt{3}}{12m^2+12m}$ and $\frac{\sqrt{3}}{12m^2}$. A significant conclusion of this analysis is that the length of the interval of admissibility, associated with a periodic sequence from the family considered here, declines with increasing N (the length of the generating string) much more rapidly than for the families of periodic sequences previously considered in Sections 5.1 and 5.2 : as $\frac{1}{N^2}$ compared with $\frac{1}{N}$.

Chapter Summary

- Several infinite families of admissible periodic sequences are exhibited; they are obtained as specific solutions of the Chua-Lin inequalities.
- A key step in the analytic determination of the intervals of admissibility is the identification of a range of θ values where a single function f_r , the subject of the r^{th} Chua-Lin inequality, dominates in magnitude all the others.
- The following infinite families of strings generate admissible periodic sequences over the associated intervals :
 - Strings $+0 \dots 0$ of lengths $N = 1, 5, 9, \dots$ for $\theta \in \left(\frac{\pi}{2}, \frac{(N+1)\pi}{2N} \right)$.
 - Strings $+0 \dots 0$ of lengths $N = 3, 7, 11, \dots$ for $\theta \in \left(\frac{(N-1)\pi}{2N}, \frac{\pi}{2} \right)$.
 - Strings $+ - 0 \dots 0$ of lengths $N = 1, 2, 3, \dots$ for $\theta \in \left(0, \frac{\pi}{N} \right)$.

The interval of admissibility given in each case is maximal, and there is no further interval of admissibility located elsewhere.

- Each of the strings $(+-)^{3m-1} 0^{6m-1}$, of lengths $N = 12m - 3$, $m = 1, 2, 3, \dots$, generates an admissible periodic sequence throughout an interval having $\frac{\pi}{3}$ as left end-point. However the right end-point is not a simple rational multiple of π . Close upper and lower bounds for the right end-point (derived via polynomial approximation of the relevant trigonometric polynomials) show that the decrease in size of the intervals of admissibility with increasing string length N is more rapid than for the families previously examined : the interval lengths diminish as $\frac{1}{N^2}$ compared to $\frac{1}{N}$.
- Related infinite families of admissible periodic sequences are deduced by means of the symmetry results described in Chapter 2.



**Dynamical Behaviour of Digital Filters subject to
2's Complement Arithmetic Nonlinearity**

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Chapter 6

Admissible Periodic Sequences

Generated by the Strings $+00+0\dots 0$

Each of the families of admissible periodic sequences we investigated in the previous chapter was derived from a succession of strings sharing a common pattern and observed amongst the results of our computer searches. We adopt a different approach here. The family of periodic sequences we now investigate arose by formulating a simple pattern for the generating strings, and asking whether admissible periodic sequences of this type could occur. On the one hand the simplicity of the generating strings suggested that the family should be amenable to theoretical analysis, but on the other hand a cursory examination of the first few cases suggested there was nothing here at all. As it has turned out both suppositions were wrong : the analysis proved to be quite difficult but the results that were uncovered are extremely interesting.

6.1 The strings $+00+0\dots 0$ as generators for periodic sequences

The sequences we investigate are generated by the strings of the form $+0\dots 0+0\dots 0$, where each has two $+$ digits and the remainder of the string consists of zeros. In fact this scheme turns out to be too general a formulation to study in one go, and we will restrict even

further the form of the strings we consider. We already know the answer to the question of admissibility when the two $+$'s are placed consecutively, for then the generating string is $++0\dots 0$, and we determined the intervals of admissibility for this family in Section 5.2. The next case to consider is when the two $+$'s are separated by one zero, i.e. strings of the form $+0+0\dots 0$. But there can be no admissible periodic sequences of this type, because as was shown in Chapter 3 the string $+0+$ may never appear in any admissible sequence. Indeed we saw that any string $+0\dots 0+$ of odd length may not be contained within an admissible sequence. Thus the first new type we encounter are periodic sequences generated by a string $+00+0\dots 0$, where the $+$ digits are separated by two zeros, and the second run of zeros is necessarily of even length. It is these periodic sequences which we study now.

If we consider the substrings which will appear in a periodic sequence generated by $+00+0\dots 0$, we can impose some preliminary restrictions on the ranges of θ values for which such a sequence can be admissible. The sequence must contain the substrings :

- (i) $+00+$, and this may only appear in an admissible sequence when $\frac{\pi}{3} < \theta < \frac{\pi}{2}$ or $\frac{3\pi}{4} < \theta < \pi$;
- (ii) $0+0$, which may only appear in an admissible sequence when $\frac{\pi}{3} < \theta < \frac{2\pi}{3}$.

(See Chapter 3, Sections 3.1 and 3.9.5.) Thus a periodic sequence generated by the string $+00+0\dots 0$ can only be admissible when $\frac{\pi}{3} < \theta < \frac{\pi}{2}$, and we can restrict our attention to this range of θ values.

At this point it is worth asking the question whether a sequence generated by a string $+00+0\dots 0$ can ever be admissible. At first sight it looks doubtful : the computer searches across the full range of θ values produced no sequences of this form. Likewise, for periodic sequences of small period, direct searches done by plotting graphs of the associated trigonometric functions and looking for intervals where the Chua-Lin inequalities might hold, were entirely without result. However, as we will see in the course of this chapter, the contrary is true. There are plenty of admissible sequences of this form, in fact many more than for all the other types of sequence we have considered so far. But there is a difference, and this accounts for the failure of our empirical work, the intervals of admissibility

are tiny by comparison with those we have encountered with the families of periodic sequences previously investigated.

6.2 A search for intervals of admissibility

For the families of sequences studied in the previous chapter extensive computer evidence enabled us to make plausible conjectures about intervals of admissibility, providing a good starting point for their theoretical investigation. But for the sequences we choose to study here there is no comparable material, indeed we know from the computer results no interval of admissibility for a sequence generated by any string $+00+0 \dots 0$. Accordingly we judge that our first task is to undertake an empirical investigation of the relevant Chua-Lin inequalities to discover, if possible, intervals of admissibility for some of these sequences.

Determining the functions f_r which appear in the Chua-Lin inequalities is not difficult for a periodic sequence generated by a string $+00+0 \dots 0$ of length N (necessarily even). Such a string has only two non-zero digits, so each function is the sum of two cosine terms. Following the approach used in Chapter 5, we may obtain the various functions f_r if we first write the string below the arguments of the appropriate cosine functions :

$$\begin{array}{cccccccc} \frac{N}{2}\theta & (\frac{N}{2}-1)\theta & (\frac{N}{2}-2)\theta & (\frac{N}{2}-3)\theta & (\frac{N}{2}-4)\theta & \dots & (\frac{N}{2}-2)\theta & (\frac{N}{2}-1)\theta \\ + & 0 & 0 & + & 0 & \dots & 0 & 0 \end{array}$$

and then cycle the string into each of its N successive alignments. The function which corresponds to the initial position of the string, displayed above, is $\cos \frac{N}{2}\theta + \cos (\frac{N}{2}-3)\theta$, and when the string is cycled to the right we will first obtain the set of functions

$$\cos(r+2)\theta + \cos(r-1)\theta \quad \text{for } r = 0, 1, \dots, \frac{N}{2}-2.$$

It is enough to include here only these $(\frac{N}{2}-1)$ functions, because after cycling beyond mid-way we produce repetitions of functions already obtained. Finally there are two alignments

for the string which arise after "wrap-around", both produce the function

$$\cos\left(\frac{N}{2}-1\right)\theta + \cos\left(\frac{N}{2}-2\right)\theta.$$

By analogy with the scheme adopted in Section 5.3 we let ψ denote this wrap-around function and the earlier set of functions are the ϕ_r 's.

To summarise, the functions f_r in the Chua-Lin inequalities for a periodic sequence generated by a string $+00+0\dots 0$ of length N are

$$\begin{aligned}\phi_r(\theta) &= \cos(r+2)\theta + \cos(r-1)\theta \quad \text{for } r = 0, 1, \dots, \frac{N}{2}-2, \\ &= 2 \cos\left(r+\frac{1}{2}\right)\theta \cos\frac{3}{2}\theta,\end{aligned}\tag{6.1}$$

$$\begin{aligned}\psi(\theta) &= \cos\left(\frac{N}{2}-1\right)\theta + \cos\left(\frac{N}{2}-2\right)\theta \\ &= 2 \cos\frac{1}{2}(N-3)\theta \cos\frac{1}{2}\theta,\end{aligned}\tag{6.2}$$

and the task of seeking intervals of admissibility for such a sequence is to find for which θ all of $\phi_0, \phi_1, \dots, \phi_{\frac{N}{2}-2}$ and ψ are contained within the envelope of $\pm g$; of course $g(\theta) = \sin\frac{N}{2}\theta \sin\theta$, as previously.

The starting point in our search for intervals of admissibility is to use graphical techniques to find intervals for short length strings. However, the most obvious approach, that of plotting the functions f_r along with the envelope of g and $-g$, is of little use for other than the very shortest generating strings. Figure 6.1 shows such a plot for the string $+00+0\dots 0$ of length 22, a string which we will later show to be the shortest one of this type to generate an admissible periodic sequence. The outcome is a complicated tangle of trigonometric curves, and it is not at all clear how to read off any interval for θ where the Chua-Lin inequalities are obeyed, i.e. where all the ϕ_r 's and ψ lie within the envelope of $\pm g$.

A more successful tactic is not to plot the graphs of *all* the functions, but instead to define (and then plot) a single function which is the maximum of the absolute values of all

the ϕ_r 's and ψ . We denote this function by f , so that

$$f = \max \left(|\phi_0|, |\phi_1|, \dots, |\phi_{N-2}|, |\psi| \right). \quad (6.3)$$

Values of θ where the Chua-Lin inequalities are satisfied can then be found by inspecting a plot of f and $|g|$. Such a plot is shown in Figure 6.2, once again for the case $N = 22$. It is now much easier to identify ranges of θ values where the two curves are close (and consequently where an interval of admissibility may be concealed).

Referring to Figure 6.2, the place on the plot which might correspond to an interval of admissibility is where the graph of f appears to touch the graph of $|g|$, a short way to the right of $\theta = 1$. By magnifying this part of the graph (see Figure 6.3), we can determine more precisely the location (just above $\theta = 1.18$) and examine more carefully what is happening there. Further magnification (Figure 6.4) reveals that the graph of f crosses and lies below the graph of $|g|$, so there is an interval of admissibility, albeit a very small one.

Thorough searching, by means of this graphical method, was conducted for the first few strings of the type $+00+0 \dots 0$; the lengths $N = 8, 10, 12, \dots, 42$ were investigated in this way. Intervals of admissibility were found to exist for only three of these lengths, namely $N = 22, 32$ and 42 ; in these cases intervals of admissibility were approximately determined as follows :

| String Length (N) | Approximate Interval of Admissibility |
|--------------------------|--|
| 22 | (1.1826, 1.1831) |
| 32 | (1.2123, 1.2125) |
| 42 | (1.2250, 1.2251) |

Plotting the graphs becomes too time consuming to permit the investigation of longer strings. Although this technique has demonstrated that there are admissible sequences arising from the strings $+00+0 \dots 0$, the evidence gathered so far is insufficient as a basis for general conjectures. (One might suppose from this evidence that the lengths $N = 10m + 12$ correspond to an infinite family of admissible periodic sequences, but we shall see that this

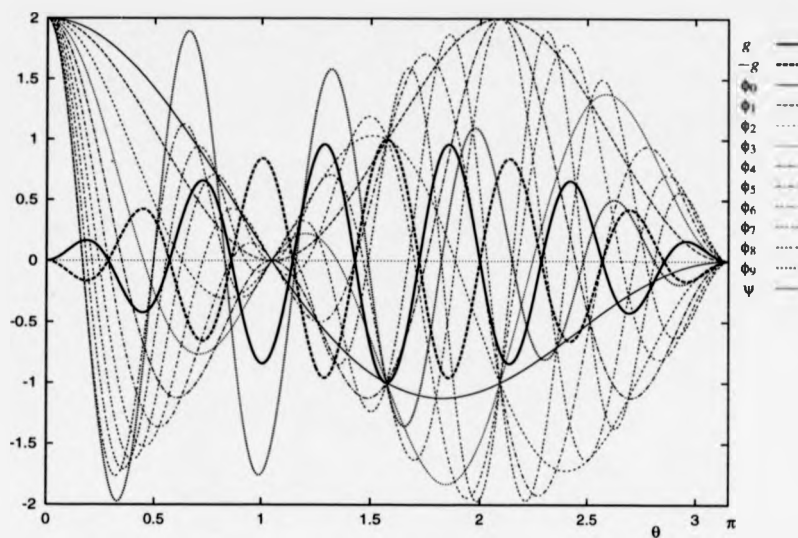


Figure 6.1: Plot of $\phi_0, \phi_1, \dots, \phi_9, \psi$ and $\pm g$ for the string $+00+0\ldots 0$ of length $N = 22$. Seeking an interval of admissibility is difficult with such a tangle of functions.

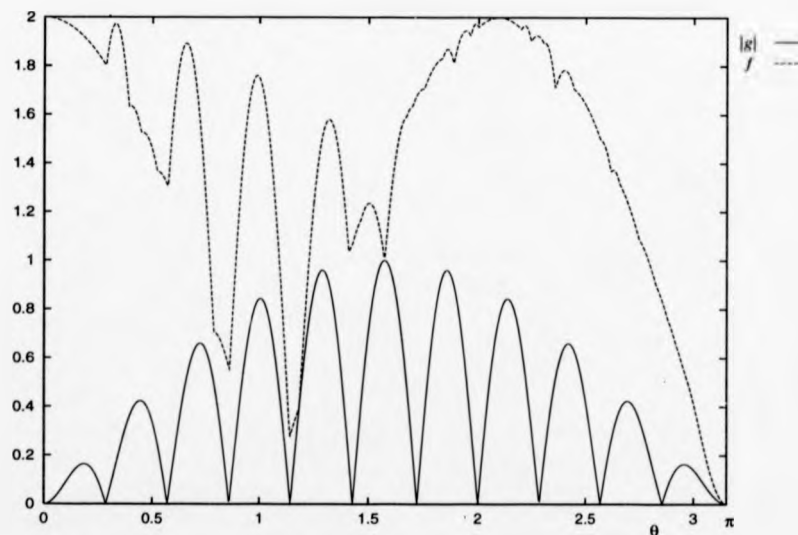


Figure 6.2: Plot of $f = \max(|\phi_0|, |\phi_1|, \dots, |\phi_9|, |\psi|)$ and $|g|$ for $N = 22$. Identifying where there might be an interval of admissibility is now an easier task.

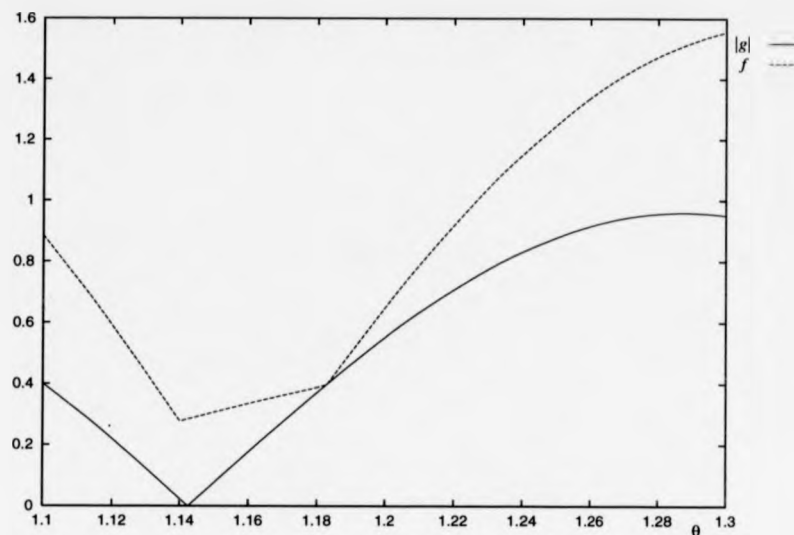


Figure 6.3: An enlargement of the part of Figure 6.2 where an interval of admissibility may be concealed.

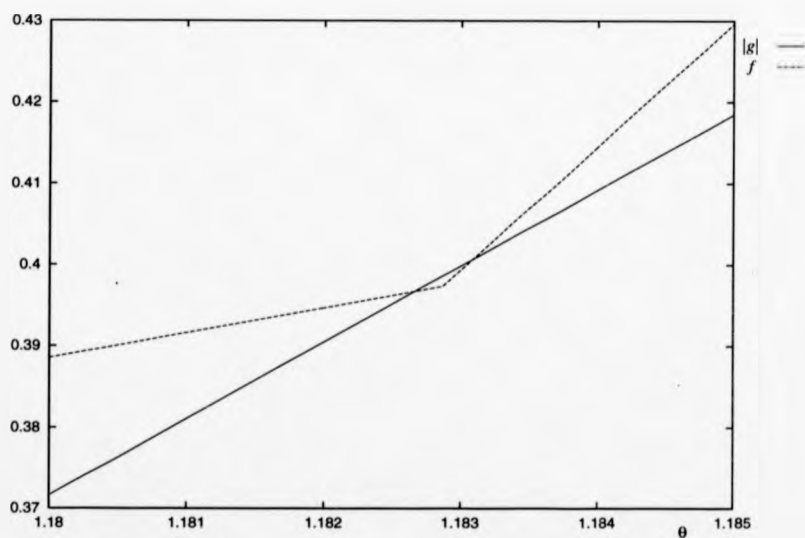


Figure 6.4: A further magnification of Figure 6.2, revealing an interval where f lies below $|g|$, so an interval of admissibility.

is not so, indeed the string with $N = 92$ generates a periodic sequence having no interval of admissibility. The actual situation for these strings is much more subtle.)

An automated search procedure for intervals of admissibility is needed, effective for strings of length considerably greater than those we have been able to work with here.

6.3 Automating the search for intervals of admissibility

The task of computing intervals of admissibility for the periodic sequence generated by a string $+00+0 \dots 0$ is fundamentally different from conducting the computer searches, at a fixed parameter value, described in Chapter 4. There the challenge was in cutting down the number of strings to test; actually determining whether a given string generated an admissible periodic sequence at the specified parameter value θ was the simple part of that task. Here the problem arises in reverse: the string is known, but we seek information about the *range of values* of θ for which it generates an admissible periodic sequence.

With no other means of deciding whether a string generates an admissible periodic sequence, we are still bound to use the criterion for admissibility (Proposition 2.1), or the Chua-Lin inequalities, as our basic test. Consequently our scheme for detecting intervals of admissibility must operate by sampling values of θ at which to invoke the test. From the little that we learned above about the size of the intervals of admissibility for the periodic sequences generated by strings $+00+0 \dots 0$, it is clear that choosing appropriate sampling points is not a straightforward task. Further evidence for this is provided by Figures 6.5, 6.6 and 6.7, where we see that for $N = 26$ the graph of f approaches $|g|$ as for $N = 22$, but that under sufficient magnification this time the graphs narrowly avoid crossing and there is no interval of admissibility.

As a first attempt at an automated strategy, we might consider the selection of a uniformly spaced sample of θ values through $(\frac{\pi}{3}, \frac{\pi}{2})$, invoking the test for admissibility at each of these values. Provided the sampling points are sufficiently narrowly spaced, the intervals of admissibility will be correctly identified. Of course, it is only necessary to locate one value of θ within an interval for which the string generates an admissible periodic sequence;

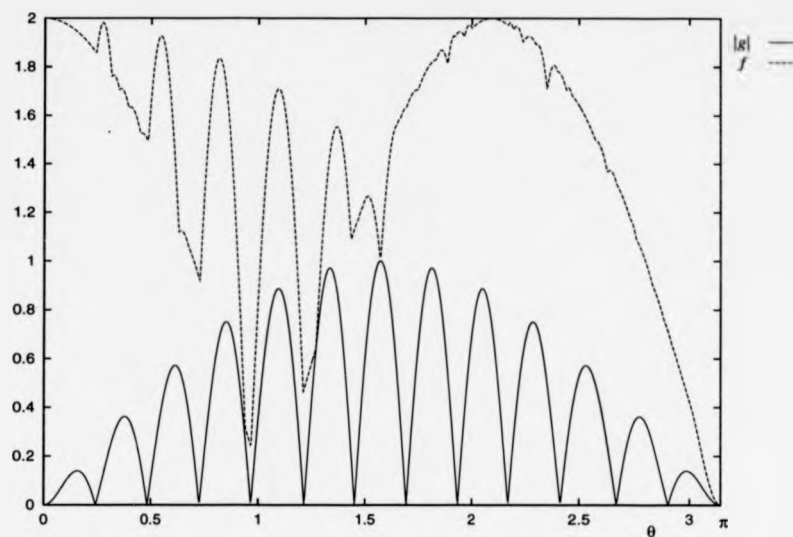


Figure 6.5: Plot of the functions f and $|g|$ for the string of length $N = 26$. Unlike the case $N = 22$, there is no interval of admissibility, though the function f does come very close to $|g|$.

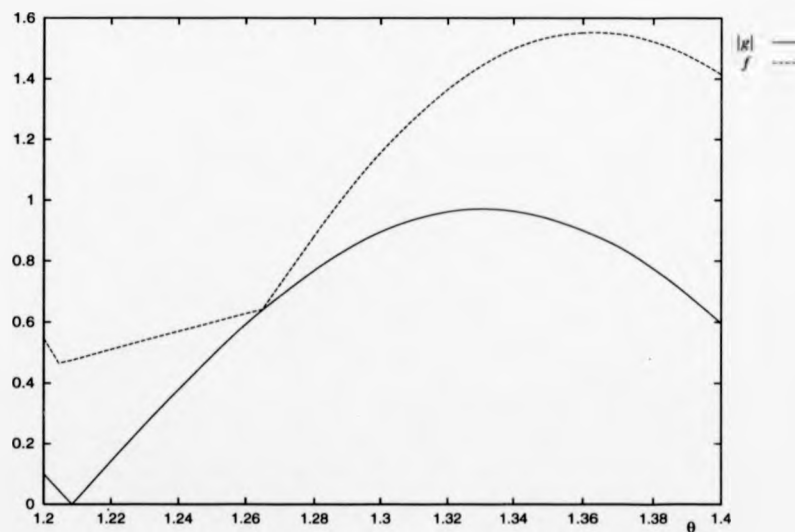


Figure 6.6: An enlargement of part of Figure 6.5. For this degree of magnification, the plot looks virtually identical to Figure 6.3.

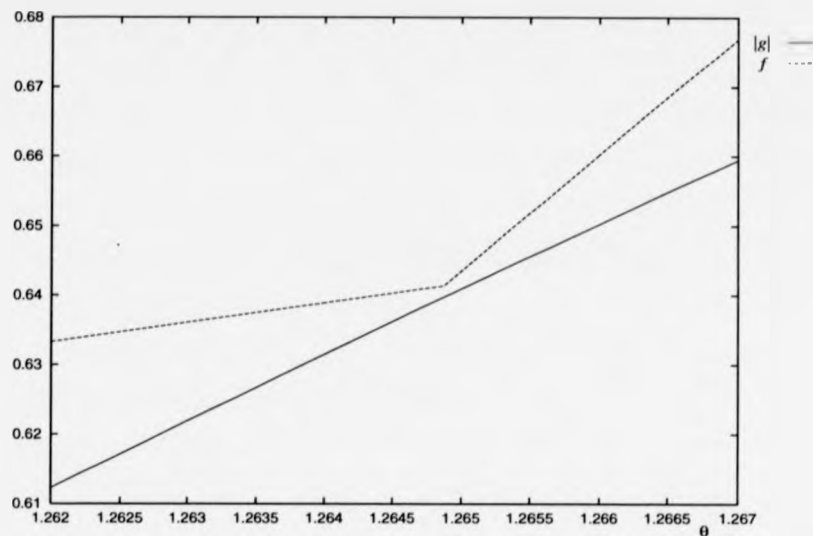


Figure 6.7: A further enlargement of Figure 6.5. Only at this level of magnification is it apparent that there is no interval of admissibility.

the end-points of the interval can then be found by the bisection method of Section 4.2.3. What limits the practicality of this approach is the enormous number of sampling points required. Even for the string length $N = 42$ (modest in itself), to detect the interval of admissibility, the separation $\delta\theta$ between samples would have to satisfy $\delta\theta < 0.0001$ (cf. the table in Section 6.2), which would imply in excess of 5000 calls to the test for admissibility.

Instead we let the values of the functions f and $|g|$ be a guide to the sampling frequency needed. Rather than trying a direct search for the small intervals where f lies below $|g|$, we approach the problem in stages, attempting first to eliminate from consideration parts of the parameter range within which f is well above $|g|$. This latter task is easier than determining precisely where f lies below $|g|$. A relatively coarse separation of sampling points suffices, for the reason that $f - |g|$ is Lipschitz, with constant $O(N)$; if f lies well above $|g|$ at a value θ we can be sure that this persists for some way to either side of θ . Once the majority of the parameter range is eliminated, the method is that within the remainder, where any intervals of admissibility are to be found, we can afford to sample at a narrower spacing.

By repeating the process, the parts of the parameter range possibly concealing an interval of admissibility are selected for closer and closer inspection.

The steps of this adaptive search strategy are as follows:

1. Subdivide the current interval (a, b) , under scrutiny, with a constant number (e.g. 1000 when $N < 1000$) of equally spaced θ values. At all points of subdivision evaluate and store the values of $f - |g|$. Accumulate and store also their maximum and minimum.
2. With a threshold determined from the maximum and minimum recorded in step 1, test each point of subdivision to ascertain which subintervals of the interval (a, b) are to be eliminated and which are to be examined more closely.
3. Recursively re-apply the adaptive search to each of the subintervals referred from step 2 whenever the length is greater than a pre-specified tolerance.

The repeated selection of those parts of the range where $f - |g|$ is smallest means that each of the intervals remaining at the end of the adaptive search is located around a local minimum of $f - |g|$. A local minimum does not necessarily imply that $f < |g|$, so not all of the final intervals need correspond to an interval of admissibility. But the tolerance is set to so small a value as to ensure that the intervals remaining at the end of the adaptive search are tiny, each lying wholly inside or wholly outside of an interval of admissibility. To discriminate between these possibilities, it is a simple matter to check whether the maximum value of $f - |g|$ is negative.

An implementation of the search procedure is given in Listing 6.1. There we use a threshold of $\frac{1}{10}(\max(f - |g|) - \min(f - |g|))$, rejecting any part of the range where the discrepancy between f and $|g|$ is more than one-tenth of its maximum. Table 6.1 shows an example of the adaptive search in operation; in it we list the intervals (a, b) for each of the recursive calls made whilst finding where the periodic sequence generated by the string $+00+0\dots 0$, of length $N = 22$, is admissible. Notice that as a result of the initial call, for which the interval supplied is $(0, \pi)$, three subintervals are identified for closer

find_range (a, b)

Find intervals of admissibility within the range $a < \theta < b$ for some periodic sequence.

```

min = 999.9; max = -999.9;

for (i=0; i<npoints; i++) {
    th = (b-a) * i / npoints + a;
    disc[i] = f(th) - abs(g(th));
    if (disc[i] < min) min = disc[i];
    if (disc[i] > max) max = disc[i];
}

threshold = (max - min) / 10;

i=0;
while (i<npoints) {
    if (disc[i] < min + threshold) {
        new_a = (b-a) * i / npoints + a;
        while ((disc[i] < min + threshold) && (i<npoints))
            i = i + 1;
        new_b = (b-a) * i / npoints + a;

        if (new_b - new_a > tolerance)
            ➤ Recursively call find_range with the smaller range (new_a, new_b).
        else if (max < 0)
            ➤ The value  $\theta = \frac{1}{2}(a + b)$  lies within an interval of admissibility. Calculate this interval using bisection and report it.
    }
    i += 1;
}

```

Listing 6.1: Code to implement the adaptive search strategy to find the intervals of admissibility for which a string $+00+0\dots 0$ generates an admissible periodic sequence.

examination, one around each of 1.18, $\frac{\pi}{2}$ and π . Reference to the plot of f and $|g|$ for this string, Figure 6.2, confirms that these are the three locations where the two curves come close. Each of these intervals is narrowed by a succession of recursive calls (the three groups of rows in the table), each call rejecting those parts of the current interval where $f - |g|$ exceeds one-tenth of its maximum discrepancy. In the final row of each group the intervals are shorter than the tolerance (which for the purpose of this example has been set to 10^{-5} , though in practise we would use 10^{-12}). The value of "max" is then inspected to determine whether any of these last intervals comes within an interval of admissibility. The procedure correctly identifies only the vicinity near $\theta = 1.18$ as the location for an interval of admissibility; at the two other possibilities, near $\frac{\pi}{2}$ and π , the graph of f approaches that of $|g|$ (indeed the two curves even touch at $\theta = \frac{\pi}{2}$ and $\theta = \pi$) but it does not cross and lie below $|g|$ in order to provide an interval of admissibility. As a final point about this example we remark that none of the points of the initial sample, taken through the full interval $(0, \pi)$, fell within the interval of admissibility (we know this because the value of "min" in the first row is positive). This behaviour is typical of the mode of operation of the adaptive search, where the initial sample will mostly be very much more coarse than would be needed to isolate the intervals of admissibility directly.

No part of the adaptive search strategy described above depends particularly on the string having to belong to the specific family $+00+0\dots 0$, and indeed it can be employed without alteration to provide the routine required for our comprehensive search in Chapter 4, described in Section 4.3. (This said, certain refinements are desirable for efficiency. In particular the proportion of the interval rejected is typically so great that the number of sampling points can be reduced as the depth of recursion increases, though of course not so much as to undermine the effect of the adaptive strategy. Secondly an extra termination condition may be added to avoid the deep recursive calls that arise when $f - |g|$ is virtually flat across the range under consideration.) We remark that when the adaptive search is used for this application, the final tiny interval identifying the location of a local minimum of $f - |g|$ may be of interest in itself (beyond its key purpose of providing a value of θ within the interval of admissibility to pass to the bisection procedure). For some strings

| End-points of Range Under Consideration | | Minimum and Maximum Values of $f - g $ at Points Sampled in Current Range | |
|--|-------------|---|-----------------|
| a | b | min | max |
| 0.0 | 3.141592741 | 4.808253453e-07 | 2.0 |
| 1.156106129 | 1.218937984 | -0.001245285356 | 0.2037096024 |
| 1.179730906 | 1.186516746 | -0.001204274291 | 0.01979969225 |
| 1.182526672 | 1.183259543 | -0.001259422289 | 0.0009757881652 |
| 1.182831547 | 1.182913628 | -0.001268687555 | -0.001010380228 |
| 1.182864379 | 1.182873408 | -0.001269452925 | -0.001240512433 |
| 1.545663629 | 1.589645927 | 0.0002769593647 | 0.2150659852 |
| 1.567742742 | 1.572844689 | 8.342054134e-05 | 0.02242490566 |
| 1.570477386 | 1.571007988 | 4.057067965e-06 | 0.002282828014 |
| 1.570766033 | 1.570821216 | 6.285656919e-07 | 0.0002677297875 |
| 1.570792521 | 1.570799143 | 2.180494531e-07 | 3.025083388e-05 |
| 3.02535381 | 3.141592741 | 9.450924119e-07 | 0.1988131953 |
| 3.107650973 | 3.141592741 | 3.221568526e-07 | 0.0198754685 |
| 3.131002909 | 3.141592741 | 1.253427783e-07 | 0.001959785201 |
| 3.138288714 | 3.141592741 | 1.900308252e-08 | 0.0001910046434 |
| 3.140568492 | 3.141592741 | 1.804692596e-09 | 1.835561661e-05 |
| 3.141275224 | 3.141592741 | 1.668482428e-10 | 1.763325133e-06 |
| 3.141494311 | 3.141592741 | 1.407689164e-11 | 1.692480445e-07 |
| 3.141562228 | 3.141592741 | 8.294671141e-13 | 1.620043803e-08 |
| 3.141583282 | 3.141592741 | 8.993349448e-16 | 1.537012279e-09 |

Table 6.1: An illustration of the adaptive search in progress : the series of parameter intervals (a, b) examined in the recursive calls to `find_range` for the string $+00+000000000000000000$.

that location coincides with a simple rational multiple of π when neither end-point of the interval of admissibility does. The association of a value of θ which is a rational multiple of π with an interval of admissibility is likely to aid the theoretical study of these strings; for that reason these values have been included in the results of the comprehensive search, listed in Table 3 of Volume II.

Using our adaptive search with initial search interval $(\frac{\pi}{3}, \frac{\pi}{2})$ we have been able to locate successfully the intervals of admissibility for periodic sequences generated by strings $+00+0 \dots 0$. This we did for strings including as many as 1000 digits, sufficient to gain some idea of the complex nature of the issue of admissibility for these sequences. The full list of intervals appears as Table 5 in Volume II; here we include Table 6.2 as an abridged version. Figure 6.8 presents a graphical representation of the intervals; it shows the intervals of admissibility, plotted in the vertical direction, against the string length N . The majority of the intervals appear in the figure as a single point, the reason being, of course,

| String Length (N) | Interval of Admissibility | Functions Determining End-points | |
|--------------------------|-----------------------------|----------------------------------|--------|
| | | Left | Right |
| 22 | (1.182666294 , 1.183086199) | ϕ_2 | ψ |
| 32 | (1.212299551 , 1.212523743) | ϕ_2 | ψ |
| 38 | (1.180413259 , 1.180512148) | ϕ_{10} | ψ |
| 42 | (1.225019586 , 1.225134087) | ϕ_2 | ψ |
| 52 | (1.232075937 , 1.232135446) | ϕ_2 | ψ |
| 54 | (1.179647264 , 1.179690319) | ϕ_{18} | ψ |
| 58 | (1.210322576 , 1.210371959) | ϕ_{15} | ψ |
| 62 | (1.236559688 , 1.236589651) | ϕ_2 | ψ |
| 70 | (1.179261870 , 1.179285837) | ϕ_{26} | ψ |
| 72 | (1.239660020 , 1.239673030) | ϕ_2 | ψ |
| 78 | (1.223386135 , 1.223408019) | ϕ_{20} | ψ |
| 82 | (1.241931346 , 1.241934135) | ϕ_2 | ψ |
| 84 | (1.209652800 , 1.209673765) | ϕ_{28} | ψ |
| 86 | (1.179029921 , 1.179045164) | ϕ_{34} | ψ |
| 88 | (1.227503582 , 1.227510886) | ϕ_{20} | ψ |
| 98 | (1.230703259 , 1.230711695) | ϕ_{25} | ψ |
| 102 | (1.178875004 , 1.178885545) | ϕ_{42} | ψ |

Table 6.2: Results from the computer search for admissible periodic sequences generated by strings $+00+0\dots 0$. The full list, for string length $N \leq 1000$, appears as Table 5 of Volume II.

the very small size of the intervals of admissibility. (Two of the intervals at the extreme left of the plot, where the string lengths are shortest, do have a noticeable vertical extent.)

Two conclusions are readily drawn from Figure 6.8. Firstly we see that contrary to our initial doubts, there are very many instances of strings with the pattern $+00+0\dots 0$ generating admissible periodic sequences. And secondly, we find that the locations of the intervals follow a much more complicated pattern (but certainly not a random one) than for any of the types of sequence studied in Chapter 5. Indeed, to describe collectively the strings of the form $+00+0\dots 0$ as a single family would seem to be wrong, because in Figure 6.8 we can identify several virtually flat progressions of equally spaced intervals across the plot, each of which potentially forms an infinite family of admissible periodic sequences.

In the next section we follow up and substantiate our computer search results with an analytical confirmation for the existence of one of these families. Thereafter we return, in Section 6.5, to discuss in more detail the involved patterns present in Figure 6.8, that is, the complex distribution of the locations of the many intervals of admissibility.

6.4 An analytical investigation of a family of periodic sequences generated by strings $+00+0\ldots 0$

In this section we develop mathematical techniques that enable us to prove, for the members of an infinite family of periodic sequences generated by strings $+00+0\ldots 0$ with regularly increasing string length, the existence of intervals of admissibility, and also to describe their location and size. The results from our computer search seem to indicate the beginnings of many such families, and although we focus our attention here on a single one of these, we believe the methods presented will be equally applicable to other of the families. The most convenient family to select for our analysis is that with the least increase in length of generating string between successive members; this is the family where we have the largest amount of computational data to guide us and confirm our conclusions. The family in question is the one corresponding to the horizontal progression of points nearest the bottom of Figure 6.8. The first periodic sequence of this family is generated by a string of length $N = 22$, and for subsequent sequences the string length is stepped up by 16.

We express the string length as $N = 16m + 6$ for $m = 1, 2, 3, \dots$, and work with the notation introduced in Section 6.2 for the functions f_r appearing in the Chua-Lin inequalities, where

$$\begin{aligned}\phi_r(\theta) &= \cos(r+2)\theta + \cos(r-1)\theta \quad \text{for } r = 0, 1, \dots, 8m+1 \\ &= 2 \cos\left(r + \frac{1}{2}\right)\theta \cos \frac{3}{2}\theta\end{aligned}\tag{6.4}$$

$$\begin{aligned}\psi(\theta) &= \cos(8m+2)\theta + \cos(8m+1)\theta \\ &= 2 \cos\left(8m + \frac{3}{2}\right)\theta \cos \frac{1}{2}\theta.\end{aligned}\tag{6.5}$$

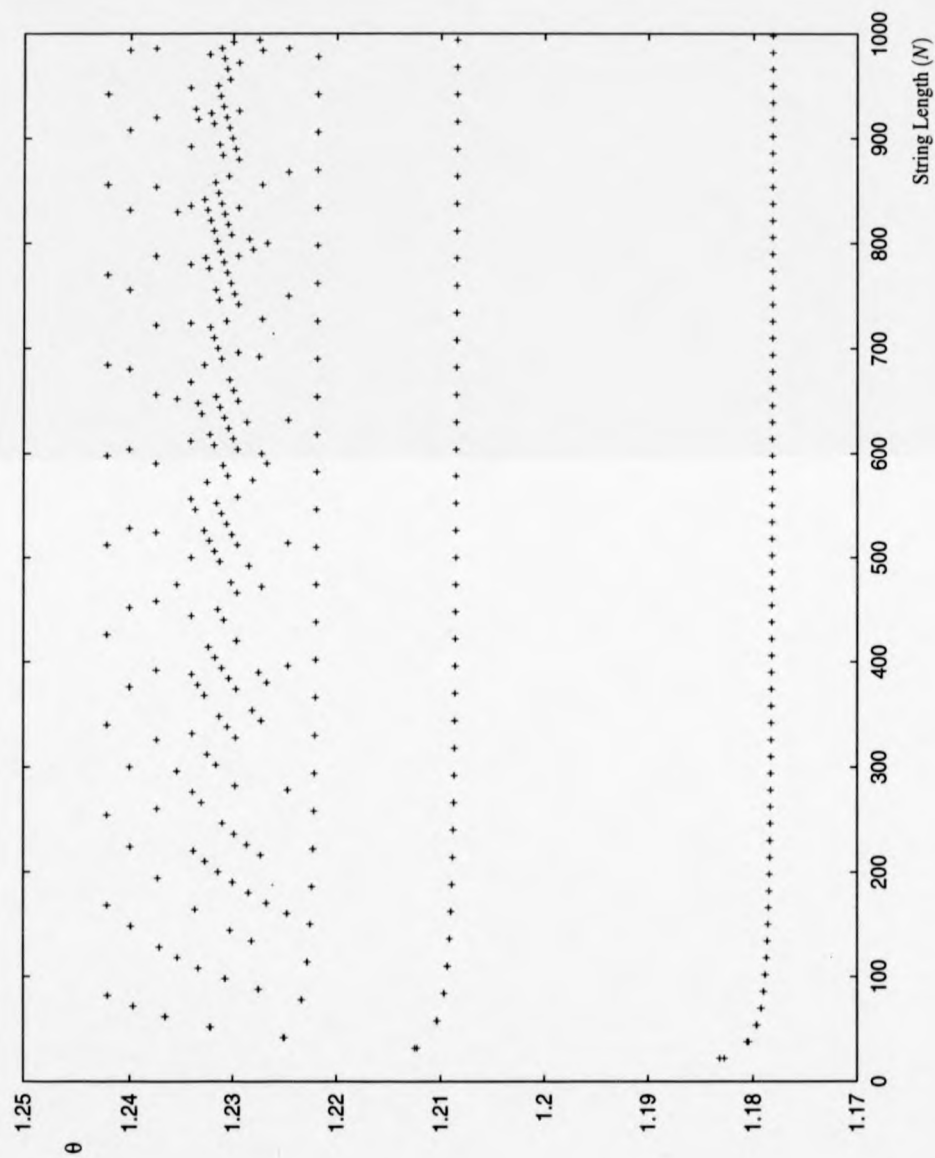


Figure 6.8: Location of the intervals of admissibility found by the adaptive computer search.

Intervals of admissibility for some of the periodic sequences from this family, as determined by the adaptive search algorithm, are shown below. The entries in the following table are taken from Table 5 in Volume II, listing all admissible sequences generated by strings $+00+0\ldots 0$, with string length N less than 1000. Shown here are the first few and the last few entries for our family (for which $N = 16m + 6$).

| m | String Length $N = 16m + 6$ | Interval of Admissibility (Computer Determined) | Functions Governing Interval End-points | |
|----------|--------------------------------|--|--|----------|
| | | | Left | Right |
| 1 | 22 | (1.182666294, 1.183086199) | ϕ_2 | ψ |
| 2 | 38 | (1.180413259, 1.180512148) | ϕ_{10} | ψ |
| 3 | 54 | (1.179647264, 1.179690319) | ϕ_{18} | ψ |
| 4 | 70 | (1.17926187, 1.179285837) | ϕ_{26} | ψ |
| 5 | 86 | (1.179029921, 1.179045164) | ϕ_{34} | ψ |
| 6 | 102 | (1.178875004, 1.178885545) | ϕ_{42} | ψ |
| \vdots | \vdots | \vdots | \vdots | \vdots |
| 61 | 982 | (1.178173968, 1.178174068) | ϕ_{482} | ψ |
| 62 | 998 | (1.178172731, 1.178172828) | ϕ_{490} | ψ |

In seeking intervals of admissibility, where the Chua-Lin inequalities hold, we are faced with a more difficult situation here than for the family of sequences with intervals at $\frac{\pi}{3}$ analysed in Section 5.3 of Chapter 5. Here, we learn from the computer data that none of the end-points of the intervals is a simple rational multiple of π . Moreover, we see from the plot of these intervals in Figure 6.9, the locations of *both* end-points shift as the string length increases. Consequently we have neither a fixed end-point nor even a moving end-point whose location is described by a simple formula in terms of N , to use as an origin for a local analysis.

Fortunately there still is a simple rational multiple of π associated with the series of intervals. Notice that in Figure 6.9, the locations of the intervals appear to approach a limiting value; inspection of the numerical values of the end-points suggests that this value is $\frac{3\pi}{8}$ (whose decimal value is approximately 1.17810). We will see later that the value $\theta = \frac{3\pi}{8}$ is to the left of each interval and that the intervals do converge to this value. Adopting a

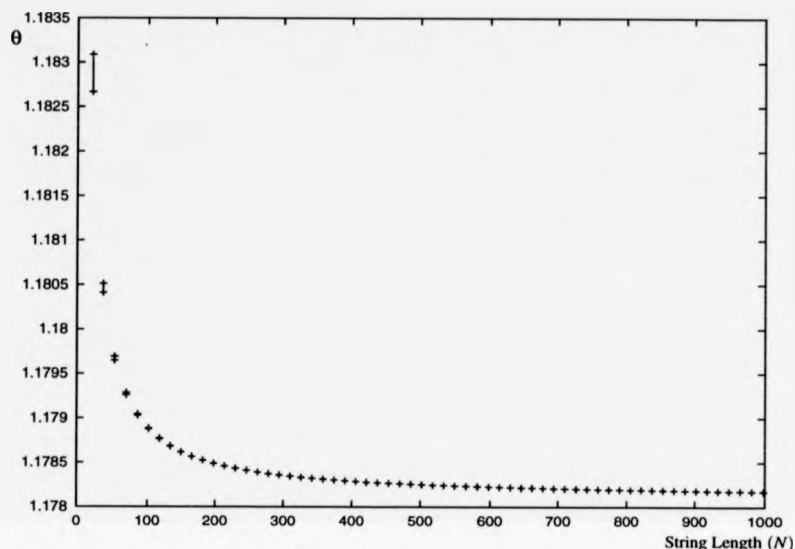


Figure 6.9: Intervals of admissibility for sequences generated by the family of strings $+00+0\dots 0$ of length $N = 16m + 6$.

similar approach to that taken in Section 5.3 of Chapter 5, when we investigated a family of sequences with intervals of admissibility at $\frac{\pi}{3}$, we seek an upper bound (dependent on m) limiting the extent of the interval of admissibility for the periodic sequence arising from the string of length $N = 16m + 6$. We shall see, by consideration of the functions ϕ_r , that a satisfactory choice for this upper bound is $\frac{6m-1}{16m-3}\pi$. When $N = 16m + 6$, the collection of functions ϕ_r is indexed from $r = 0$ to $r = \frac{N}{2} - 2 = 8m + 1$. The next proposition shows that throughout $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$ one of these functions is dominant. For our subsequent work it will be useful to remark that the length of this interval is $\frac{\pi}{8(16m-3)}$.

Proposition 6.1

Amongst the $8m + 2$ functions ϕ_r , the function ϕ_{8m-6} dominates in absolute value throughout the range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$.

Proof

The proof is by induction : assume that ϕ_{8m-6} is dominant amongst $\phi_0, \phi_1, \dots, \phi_{8m+1}$ throughout the interval $I_m = [\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$. Then we must show that ϕ_{8m+2} is dominant amongst $\phi_0, \phi_1, \dots, \phi_{8m+9}$ throughout $I_{m+1} = [\frac{3\pi}{8}, \frac{6m+5}{16m+13}\pi]$.

The first stage is to show that ϕ_{8m+2} dominates ϕ_{8m-6} in I_{m+1} . Since $I_{m+1} \subset I_m$, it will then follow from the inductive assumption that ϕ_{8m+2} dominates $\phi_0, \phi_1, \dots, \phi_{8m+1}$ there. The functions ϕ_{8m-6} and ϕ_{8m+2} are :

$$\phi_{8m-6}(\theta) = 2 \cos \frac{3}{2}\theta \cos (8m - \frac{11}{2})\theta, \quad (6.6)$$

$$\phi_{8m+2}(\theta) = 2 \cos \frac{3}{2}\theta \cos (8m + \frac{5}{2})\theta. \quad (6.7)$$

The factor $2 \cos \frac{3}{2}\theta$ is common to both, so plays no role in determining which function dominates. Thus the task is to show that of the two cosine functions, $\cos (8m + \frac{5}{2})\theta$ is larger in absolute value than $\cos (8m - \frac{11}{2})\theta$ for θ in I_{m+1} . The angles traced out by the arguments $(8m - \frac{11}{2})\theta$ and $(8m + \frac{5}{2})\theta$ are as follows:

- $(8m - \frac{11}{2})\theta$ runs from $3m\pi - \frac{33\pi}{16}$ to $3m\pi - \frac{33\pi}{16} + \frac{16m-11}{16m+13} \frac{\pi}{16}$,
- $(8m + \frac{5}{2})\theta$ runs from $3m\pi + \frac{15\pi}{16}$ to $3m\pi + \frac{15\pi}{16} + \frac{16m+5}{16m+13} \frac{\pi}{16}$.

Since we are interested only in the magnitude, not the signs of the cosine functions, we may equally consider $-\cos (8m - \frac{11}{2})\theta$ in place of $\cos (8m - \frac{11}{2})\theta$. Using the trigonometric identity $-\cos \theta = \cos (\theta + 3\pi)$, we may write

$$\phi_{8m-6}(\theta) = -2 \cos \frac{3}{2}\theta \cos ((8m - \frac{11}{2})\theta + 3\pi),$$

and the angle traced out by the argument $(8m - \frac{11}{2})\theta + 3\pi$ runs from $3m\pi + \frac{15\pi}{16}$ to $3m\pi + \frac{15\pi}{16} + \frac{16m-11}{16m+13} \frac{\pi}{16}$. Thus as θ traverses I_{m+1} , so the absolute values of $\cos (8m - \frac{11}{2})\theta$ and $\cos (8m + \frac{5}{2})\theta$ both increase from the same starting value. Appealing once more to the argument about tracing out the cosine curve at different speeds (cf. Section 5.1.1), it follows that $\cos (8m + \frac{5}{2})\theta$ dominates $\cos (8m - \frac{11}{2})\theta$, and hence that ϕ_{8m+2} dominates ϕ_{8m-6} throughout I_{m+1} .

To complete the inductive step we must additionally show that ϕ_{8m+2} dominates each of $\phi_{8m+3}, \phi_{8m+4}, \dots, \phi_{8m+9}$ throughout the range I_{m+1} . This part of the proof follows readily from the fact that the argument of the $\cos(r + \frac{1}{2})\theta$ factor in ϕ_{8m+2} lies within $\pm \frac{\pi}{16}$ of a multiple of π and this is not so for each of $\phi_{8m+3}, \phi_{8m+4}, \dots, \phi_{8m+9}$. Indeed, Table 6.3 lists the angles traced out by the arguments of the $\cos(r + \frac{1}{2})\theta$ factors for the latter functions, and it can be seen that no argument ever comes within $\pm \frac{\pi}{16}$ of a multiple of π .

| Function | Argument | Range of Argument | |
|---------------|-----------------------------|--------------------------------|--|
| | | From | To |
| ϕ_{8m+3} | $(8m + \frac{7}{2})\theta$ | $(3m+1)\pi + \frac{5\pi}{16}$ | $(3m+1)\pi + \frac{5\pi}{16} + \frac{16m+7}{16m+13} \frac{\pi}{16}$ |
| ϕ_{8m+4} | $(8m + \frac{9}{2})\theta$ | $(3m+1)\pi + \frac{11\pi}{16}$ | $(3m+1)\pi + \frac{11\pi}{16} + \frac{16m+9}{16m+13} \frac{\pi}{16}$ |
| ϕ_{8m+5} | $(8m + \frac{11}{2})\theta$ | $(3m+2)\pi + \frac{\pi}{16}$ | $(3m+2)\pi + \frac{\pi}{16} + \frac{16m+11}{16m+13} \frac{\pi}{16}$ |
| ϕ_{8m+6} | $(8m + \frac{13}{2})\theta$ | $(3m+2)\pi + \frac{7\pi}{16}$ | $(3m+2)\pi + \frac{\pi}{2}$ |
| ϕ_{8m+7} | $(8m + \frac{15}{2})\theta$ | $(3m+2)\pi + \frac{13\pi}{16}$ | $(3m+2)\pi + \frac{14\pi}{16} + \frac{2}{16m+13} \frac{\pi}{16}$ |
| ϕ_{8m+8} | $(8m + \frac{17}{2})\theta$ | $(3m+3)\pi + \frac{3\pi}{16}$ | $(3m+3)\pi + \frac{4\pi}{16} + \frac{4}{16m+13} \frac{\pi}{16}$ |
| ϕ_{8m+9} | $(8m + \frac{19}{2})\theta$ | $(3m+3)\pi + \frac{9\pi}{16}$ | $(3m+3)\pi + \frac{10\pi}{16} + \frac{6}{16m+13} \frac{\pi}{16}$ |

Table 6.3: Ranges for the arguments of the factor $\cos(r + \frac{1}{2})\theta$ as it occurs in the functions $\phi_{8m+3}, \phi_{8m+4}, \dots, \phi_{8m+9}$, for θ in I_{m+1} .

It remains to verify the initial case $m = 1$, and to do so we must show that ϕ_2 dominates $\phi_0, \phi_1, \dots, \phi_9$ for θ in $[\frac{3\pi}{8}, \frac{5\pi}{13}]$. We may reuse some of the work above: setting $m = 0$ in the expressions for the arguments of the $\cos(r + \frac{1}{2})\theta$ factors of ϕ_{8m+2} and the functions in Table 6.3 shows at once that ϕ_2 dominates $\phi_3, \phi_4, \dots, \phi_9$. Finally, in the remaining cases, where $r = 0$ and 1, the arguments of $\cos \frac{1}{2}\theta$ and $\cos \frac{3}{2}\theta$ vary as follows:

| Function | Argument | Range of Argument | |
|----------|---------------------|-------------------|---|
| | | From | To |
| ϕ_0 | $\frac{1}{2}\theta$ | $\frac{3\pi}{16}$ | $\frac{3\pi}{16} + \frac{1}{13} \frac{\pi}{16}$ |
| ϕ_1 | $\frac{3}{2}\theta$ | $\frac{9\pi}{16}$ | $\frac{9\pi}{16} + \frac{3}{13} \frac{\pi}{16}$ |

So the same reasoning applies since neither is within $\pm \frac{\pi}{16}$ of a multiple of π . \square

Figures 6.10 and 6.11 exhibit the dominance of ϕ_{8m-6} over the other functions ϕ_r in the case $m = 1$, i.e. for the first periodic sequence of the family, generated by the string $+00+0 \dots 0$ of length $N = 22$. The first plot, in Figure 6.10, corresponds to the full range of θ between $\frac{\pi}{3}$ and $\frac{\pi}{2}$. (Recall that the intervals 0 to $\frac{\pi}{3}$ and $\frac{\pi}{2}$ to π have been ruled out by more elementary considerations, given in Section 6.1.) The particular range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi] = [\frac{3\pi}{8}, \frac{5\pi}{13}]$, over which we have shown $\phi_{8m-6} = \phi_2$ dominates, is marked on the plot and has been enlarged to form the whole plotting range in Figure 6.11. Besides exhibiting the dominance of ϕ_2 in this particular instance, the plot also illustrates the reasoning employed in the proof of Proposition 6.1. The behaviour of each of the functions ϕ_r over the specified range can be matched with the arguments of the cosine functions listed in Table 6.3. For example, ϕ_6 decreases in magnitude throughout the range, rising from a negative starting value when $\theta = \frac{3\pi}{8}$ and reaching the value 0 at the right-hand end of the interval. (Its negative sign is attributable to the other factor, $\cos \frac{3}{2}\theta$, present in ϕ_6 , of course.)

The key consequence of Proposition 6.1 is that the subinterval of θ values at which the string $+00+0 \dots 0$ of length $N = 16m + 6$ generates an admissible periodic sequence is determined only by the intersections of the two functions ϕ_{8m-6} and ψ with the envelope of $\pm g$. In view of this, and to keep the notation free from clutter we shall henceforth drop the subscript on ϕ_{8m-6} and refer to this function simply as ϕ . It is also advantageous at this stage to introduce a normalisation for the functions ϕ , ψ and g that will reduce the complexity of the working to follow.

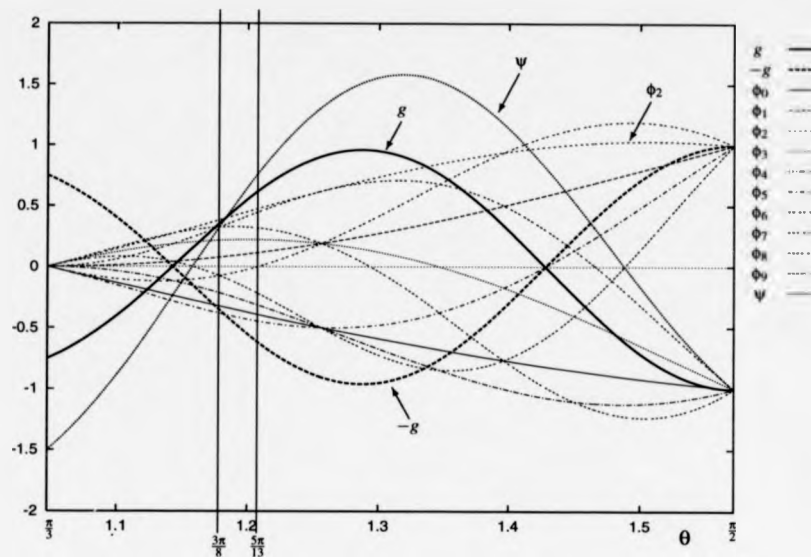
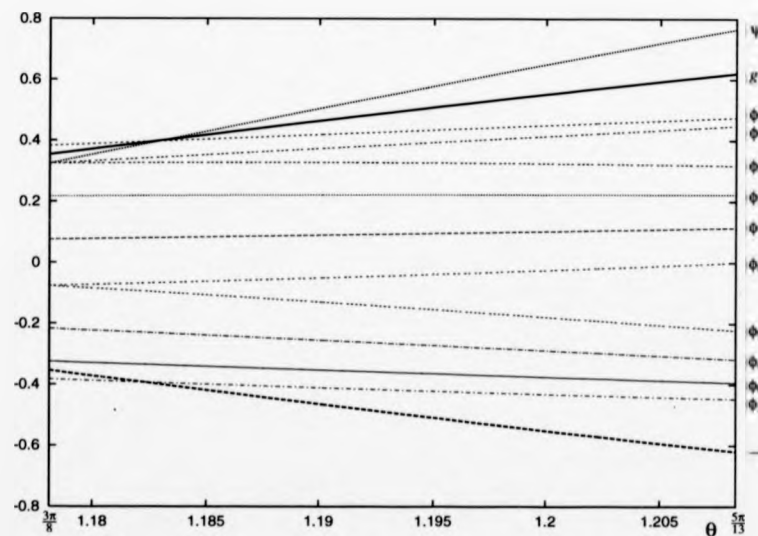
None of the functions ϕ , ψ and g change sign throughout the range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$; indeed they all share the same sign over this range. This common sign does, however, depend upon the parity of m as we now describe. We take the functions in their multiplicative forms :

$$\phi(\theta) = 2 \cos(8m - \frac{11}{2})\theta \cos \frac{3}{2}\theta,$$

$$\psi(\theta) = 2 \cos(8m + \frac{3}{2})\theta \cos \frac{1}{2}\theta,$$

$$g(\theta) = \sin(8m + 3)\theta \sin \theta.$$

Since the range in question extends from $\frac{3\pi}{8}$ to an upper value of at most $\frac{5\pi}{13}$, none of the

Figure 6.10: Plot of $\phi_0, \phi_1, \dots, \phi_9, \psi$ and $\pm g$, when $m = 1$, for θ between $\frac{\pi}{3}$ and $\frac{\pi}{2}$.Figure 6.11: Plot of $\phi_0, \phi_1, \dots, \phi_9, \psi$ and $\pm g$ as previously, but restricted to the range $[\frac{3\pi}{8}, \frac{5\pi}{13}]$.

terms $\cos \frac{3}{2}\theta$, $\cos \frac{1}{2}\theta$ or $\sin \theta$ changes sign. Exactly as in the proof of Proposition 6.1, the angle traced out by $(8m - \frac{11}{2})\theta$ runs from $(3m - 2)\pi - \frac{\pi}{16}$ to $(3m - 2)\pi - \frac{\pi}{16} + \frac{16m+1}{16m-3} \frac{\pi}{16}$, from which it follows that the sign of ϕ is $(-1)^{m+1}$. A similar investigation for $(8m + \frac{3}{2})\theta$ and $(8m + 3)\theta$ shows that the signs of both ψ and g are also $(-1)^{m+1}$ throughout the range. We thus define normalised versions of these functions by

$$\tilde{\phi} = (-1)^{m+1}\phi, \quad \tilde{\psi} = (-1)^{m+1}\psi, \quad \tilde{g} = (-1)^{m+1}g, \quad (6.8)$$

so that $\tilde{\phi}$, $\tilde{\psi}$ and \tilde{g} are all positive throughout the range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$.

Thus far we have shown that $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$ is a convenient range to work with for the reason that we may discard all but one of the functions ϕ_r when we stay within the range. However, we have yet to demonstrate that this range is an appropriate place to search for intervals of admissibility for sequences from the family we are investigating. It is this issue to which we now turn our attention.

Lemma 6.2

The values of the functions $\tilde{\phi}$, $\tilde{\psi}$ and \tilde{g} at the end-points of the range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$ are ordered as follows :

- (i) $\tilde{\psi}(\frac{3\pi}{8}) < \tilde{g}(\frac{3\pi}{8}) < \tilde{\phi}(\frac{3\pi}{8})$,
- (ii) $\tilde{\phi}(\frac{6m-1}{16m-3}\pi) < \tilde{g}(\frac{6m-1}{16m-3}\pi) < \tilde{\psi}(\frac{6m-1}{16m-3}\pi)$.

Proof

Let $\theta = \frac{3\pi}{8} + \alpha$, so that we describe $\tilde{\phi}$, $\tilde{\psi}$ and \tilde{g} relative to an origin at the left-hand end of the range. The expressions for $\tilde{\phi}$, $\tilde{\psi}$ and \tilde{g} in terms of α are :

$$\begin{aligned} \tilde{\phi}(\frac{3\pi}{8} + \alpha) &= 2(-1)^{m+1} \cos[(8m - \frac{11}{2})(\frac{3\pi}{8} + \alpha)] \cos \frac{3}{2}(\frac{3\pi}{8} + \alpha) \\ &= 2 \cos[\frac{\pi}{16} - (8m - \frac{11}{2})\alpha] \cos(\frac{7\pi}{16} - \frac{3}{2}\alpha), \end{aligned} \quad (6.9)$$

$$\begin{aligned} \tilde{\psi}(\frac{3\pi}{8} + \alpha) &= 2(-1)^{m+1} \cos[(8m + \frac{3}{2})(\frac{3\pi}{8} + \alpha)] \cos \frac{1}{2}(\frac{3\pi}{8} + \alpha) \\ &= 2(-1)^{m+1} \cos[3m\pi + \frac{9\pi}{16} + (8m + \frac{3}{2})\alpha] \cos \frac{1}{2}(\frac{3\pi}{8} + \alpha) \\ &= 2 \cos[\frac{7\pi}{16} - (8m + \frac{3}{2})\alpha] \cos \frac{1}{2}(\frac{3\pi}{8} + \alpha), \end{aligned} \quad (6.10)$$

$$\begin{aligned}
\bar{g}\left(\frac{3\pi}{8} + \alpha\right) &= (-1)^{m+1} \sin\left[(8m+3)\left(\frac{3\pi}{8} + \alpha\right)\right] \sin\left(\frac{3\pi}{8} + \alpha\right) \\
&= 2(-1)^{m+1} \sin\left[3m\pi + \frac{9\pi}{8} + (8m+3)\alpha\right] \sin\frac{1}{2}\left(\frac{3\pi}{8} + \alpha\right) \cos\frac{1}{2}\left(\frac{3\pi}{8} + \alpha\right) \\
&= 2 \sin\left[\frac{\pi}{8} + (8m+3)\alpha\right] \sin\left(\frac{3\pi}{16} + \frac{1}{2}\alpha\right) \cos\frac{1}{2}\left(\frac{3\pi}{8} + \alpha\right). \quad (6.11)
\end{aligned}$$

At the lower end of the range, $\theta = \frac{3\pi}{8}$, corresponding to $\alpha = 0$, the values taken by the functions \bar{g} , $\bar{\psi}$ and $\bar{\phi}$ are :

$$\begin{aligned}
\bar{\phi}\left(\frac{3\pi}{8}\right) &= 2 \cos \frac{\pi}{16} \cos \frac{7\pi}{16}, \\
\bar{\psi}\left(\frac{3\pi}{8}\right) &= 2 \cos \frac{7\pi}{16} \cos \frac{3\pi}{16}, \\
\bar{g}\left(\frac{3\pi}{8}\right) &= 2 \sin \frac{\pi}{8} \sin \frac{3\pi}{16} \cos \frac{3\pi}{16},
\end{aligned}$$

from which $\bar{\psi} < \bar{g} < \bar{\phi}$ at the left end-point (because it is easily checked that $\bar{\psi} < 0.32$, $0.35 < \bar{g} < 0.36$, and $\bar{\phi} > 0.38$).

The upper end of the range is $\theta = \frac{6m-1}{16m-3}\pi$, which corresponds to $\alpha = \frac{\pi}{8(16m-3)}$. With this value for α , from expressions (6.10) and (6.11) for $\bar{\psi}$ and \bar{g} , we deduce the inequalities

$$\begin{aligned}
\bar{\psi}\left(\frac{6m-1}{16m-3}\pi\right) &= 2 \cos\left(\frac{7\pi}{16} - \frac{16m+3}{16m-3} \frac{\pi}{16}\right) \cos\frac{1}{2}\left(\frac{3\pi}{8} + \alpha\right) \\
&= 2 \cos\left(\frac{6\pi}{16} - \frac{6}{16m-3} \frac{\pi}{16}\right) \cos\frac{1}{2}\left(\frac{3\pi}{8} + \alpha\right) \\
&> 2 \cos \frac{3\pi}{8} \cos\frac{1}{2}\left(\frac{3\pi}{8} + \alpha\right), \quad (6.12)
\end{aligned}$$

$$\begin{aligned}
\bar{g}\left(\frac{6m-1}{16m-3}\pi\right) &= 2 \sin\left(\frac{\pi}{8} + \frac{16m+6}{16m-3} \frac{\pi}{16}\right) \sin\left(\frac{3\pi}{16} + \frac{1}{16m-3} \frac{\pi}{16}\right) \cos\frac{1}{2}\left(\frac{3\pi}{8} + \alpha\right) \\
&= 2 \sin\left(\frac{3\pi}{16} + \frac{9}{16m-3} \frac{\pi}{16}\right) \sin\left(\frac{3\pi}{16} + \frac{1}{16m-3} \frac{\pi}{16}\right) \cos\frac{1}{2}\left(\frac{3\pi}{8} + \alpha\right) \\
&< 2 \sin\left(\frac{3\pi}{16} + \frac{9}{13} \frac{\pi}{16}\right) \sin\left(\frac{3\pi}{16} + \frac{1}{13} \frac{\pi}{16}\right) \cos\frac{1}{2}\left(\frac{3\pi}{8} + \alpha\right), \quad (6.13)
\end{aligned}$$

and, because $2 \cos \frac{3\pi}{8} > 0.76$ and $2 \sin\left(\frac{3\pi}{16} + \frac{9}{13} \frac{\pi}{16}\right) \sin\left(\frac{3\pi}{16} + \frac{1}{13} \frac{\pi}{16}\right) = 2 \sin \frac{3\pi}{13} \sin \frac{5\pi}{26} < 0.76$, we have that $\bar{\psi} > \bar{g}$ at the upper end-point. Similarly, expressions (6.9) and (6.11) for $\bar{\phi}$ and \bar{g} give

$$\begin{aligned}
\bar{\phi}\left(\frac{6m-1}{16m-3}\pi\right) &= 2 \cos\left(\frac{\pi}{16} - \frac{16m-11}{16m-3} \frac{\pi}{16}\right) \cos\left(\frac{7\pi}{16} - \frac{3}{16m-3} \frac{\pi}{16}\right) \\
&= 2 \cos\left(\frac{8}{16m-3} \frac{\pi}{16}\right) \cos\left(\frac{7\pi}{16} - \frac{3}{16m-3} \frac{\pi}{16}\right) \\
&< 2 \cos\left(\frac{7\pi}{16} - \frac{3}{13} \frac{\pi}{16}\right), \quad (6.14)
\end{aligned}$$

$$\begin{aligned}
 \bar{g}\left(\frac{6m-1}{16m-3}\pi\right) &= \sin\left(\frac{\pi}{8} + \frac{16m+6}{16m-3}\frac{\pi}{16}\right) \sin\left(\frac{3\pi}{8} + \frac{1}{16m-3}\frac{\pi}{8}\right) \\
 &= \sin\left(\frac{3\pi}{16} + \frac{9}{16m-3}\frac{\pi}{16}\right) \sin\left(\frac{3\pi}{8} + \frac{1}{16m-3}\frac{\pi}{8}\right) \\
 &> \sin\frac{3\pi}{16} \sin\frac{3\pi}{8},
 \end{aligned} \tag{6.15}$$

so that $\bar{\phi} < \bar{g}$ at the right end-point, because $2\cos\frac{11\pi}{26} < 0.48$ but $\sin\frac{3\pi}{16} \sin\frac{3\pi}{8} > 0.51$. \square

It follows from this lemma that the function ϕ must enter the envelope formed by g and $-g$ at some point in the range $\left[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi\right]$, and likewise, ψ must leave the envelope at a point in the range. This suggests the possibility of an interval of admissibility within the range, but does not guarantee it, since ψ might leave the envelope before ϕ enters, as illustrated in Figure 6.12.

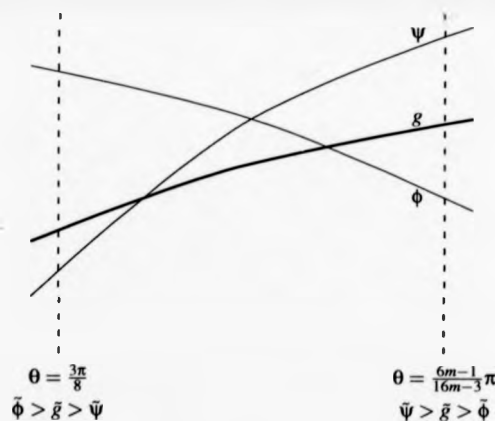


Figure 6.12: The relative ordering of the values of $\bar{\phi}$, $\bar{\psi}$ and \bar{g} at the extreme points need not guarantee an interval of admissibility within the range.

There are other troublesome difficulties not precluded because of Lemma 6.2. We cannot be sure, for instance, that the function ϕ does not cross in and out of the envelope of $\pm g$ several times in the range $\left[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi\right]$. As well, attempts to clarify the picture in general by locating points of intersection of the graphs of either $\bar{\phi}$ or $\bar{\psi}$ with \bar{g} , by seeking solutions

of the associated trigonometric equations, have been unsuccessful.

Nevertheless, Lemma 6.2 does guarantee that there is a point of intersection of the functions $\dot{\phi}$ and $\dot{\psi}$ within the interior of the range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$ and, while the precise location of any such point is not easily determined from the corresponding trigonometric equation, the knowledge alone of the existence of a point of intersection is sufficient to show that there must be an interval of admissibility here.

Proposition 6.3 (Existence of an interval of admissibility)

The string $+00+0 \dots 0$ of length $N = 16m + 6$ generates a periodic sequence admissible within an interval contained by the range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$.

Proof

The function $2\bar{g} - (\dot{\phi} + \dot{\psi})$ may be expressed as product of cosine terms as follows :

$$\begin{aligned}
 2\bar{g}(\theta) - (\dot{\phi}(\theta) + \dot{\psi}(\theta)) &= (-1)^{m+1} \{ \cos(8m+2)\theta - \cos(8m+4)\theta - \cos(8m+2)\theta \\
 &\quad - \cos(8m+1)\theta - \cos(8m-4)\theta - \cos(8m-7)\theta \} \\
 &= (-1)^m \{ \cos(8m+4)\theta + \cos(8m-4)\theta + \cos(8m+1)\theta \\
 &\quad + \cos(8m-7)\theta \} \\
 &= 2(-1)^m \{ \cos 8m\theta \cos 4\theta + \cos(8m-3)\theta \cos 4\theta \} \\
 &= 2(-1)^m \cos 4\theta \cdot \underbrace{2 \cos(8m - \frac{3}{2})\theta \cos \frac{3}{2}\theta}_{= \phi_{8m-2}(\theta)}. \tag{6.16}
 \end{aligned}$$

For any point θ with $\frac{3\pi}{8} < \theta < \frac{6m-1}{16m-3}\pi$, $\cos 4\theta > 0$ and $\cos \frac{3}{2}\theta < 0$. The sign of the remaining term, $\cos(8m - \frac{3}{2})\theta$, may be calculated by setting $\theta = \frac{3\pi}{8} + \alpha$:

$$\begin{aligned}
 \cos(8m - \frac{3}{2})\theta &= \cos[(8m - \frac{3}{2})(\frac{3\pi}{8} + \alpha)] \\
 &= \cos[3m\pi - \frac{9\pi}{16} + (8m - \frac{3}{2})\alpha] \\
 &= (-1)^{m+1} \cos[\frac{7\pi}{16} + (8m - \frac{3}{2})\alpha].
 \end{aligned}$$

Since $0 < (8m - \frac{3}{2})\alpha < \frac{\pi}{16}$, it follows that $\cos(8m - \frac{3}{2})\theta$ has the sign $(-1)^{m+1}$. Thus from the product expression (6.16), it follows that $2\bar{g} - (\dot{\phi} + \dot{\psi}) > 0$ for all θ between $\frac{3\pi}{8}$

and $\frac{6m-1}{16m-3}\pi$.

In particular, if θ^* is a point of intersection of $\tilde{\phi}$ and $\tilde{\psi}$ within the range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$, then

$$\bar{g}(\theta^*) - \tilde{\phi}(\theta^*) = \bar{g}(\theta^*) - \tilde{\psi}(\theta^*) = \frac{1}{2}(2\bar{g}(\theta^*) - (\tilde{\phi}(\theta^*) + \tilde{\psi}(\theta^*))) > 0.$$

Thus the string $+00+0 \dots 0$ of length $N = 16m + 6$ generates an admissible sequence at $\theta = \theta^*$. The continuity of \bar{g} , $\tilde{\psi}$ and $\tilde{\phi}$ ensures that the point $\theta = \theta^*$ lies in some small interval for θ throughout which the sequence is admissible. \square

With the existence of an interval of admissibility now confirmed, the next interesting questions pertain to its location and size. Although we know that an interval of admissibility lies within the range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$, this range should not be thought of as an approximation to the interval itself. This particular range was chosen for convenience when working with the functions ϕ_r in the vicinity of a suspected interval of admissibility; in fact the interval itself is very much smaller than the containing range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$. An illustration that this is so is given in Figure 6.13 where, for $m = 1$, the functions ϕ , ψ and g are plotted over the range $[\frac{3\pi}{8}, \frac{5\pi}{13}]$. The interval of admissibility (the part of the plot where both ϕ and ψ lie below g) covers only a small fraction of the range.

Because of the lack of analytic solutions for the trigonometric equations associated with determining the intersections of the graphs of ϕ and ψ with that of g , we revert to the approach adopted in our earlier work of Section 5.3, on periodic sequences with intervals of admissibility near $\frac{\pi}{3}$, and seek approximations for the end-points of the interval of admissibility. Previously we showed how to enclose the end-point between tight upper and lower bounds, by replacing the trigonometric functions with approximating polynomials (appropriately selected in order to respect inequalities for the bound). We envisage that the same procedure could be applied in the present case to show that there is a single interval of admissibility within the range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$ and to give accurate upper and lower bounds for each of its end-points. Rather than give a long exposition leading to a derivation of

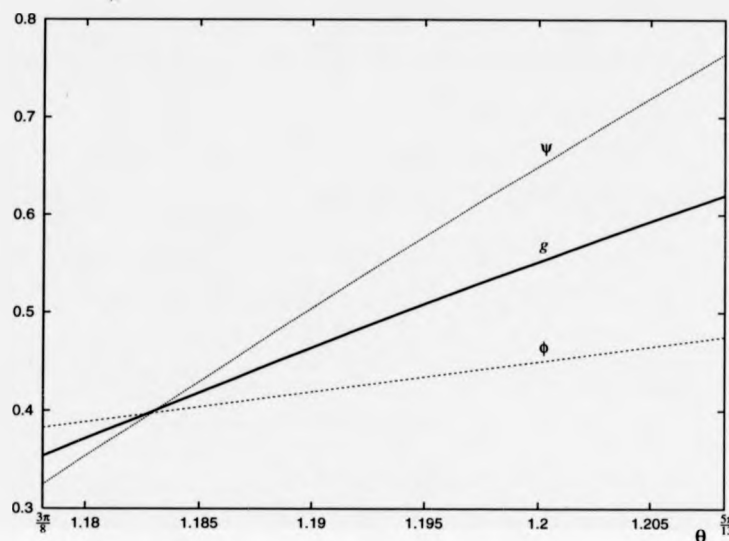


Figure 6.13: Plot, in the case $m = 1$, of ϕ , ψ and g over the range $[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi]$. On this scale the interval of admissibility is barely detectable: the length of the containing range much exceeds that of the interval of admissibility.

these bounds, we will avoid repeating ideas and techniques already presented in Chapter 5, and instead take the short-cut of obtaining direct approximations to the interval end-points under the assumption that there is a single interval of admissibility within the range.

We will search for the right end-point of the interval of admissibility as the first intersection of the graphs of ψ and g to the right of $\theta = \frac{3\pi}{8}$. Note that there is no advantage in working with the normalised versions of the functions when seeking their intersections, so now we go back to the original functions. Using the additive expressions for ψ and g , these two functions are equal when

$$\cos(8m+2)\theta + \cos(8m+1)\theta - \frac{1}{2}(\cos(8m+2)\theta - \cos(8m+4)\theta) = 0,$$

i.e. when $Q_R(\theta) = 0$, where

$$\begin{aligned} Q_R(\theta) &= 2(\psi(\theta) - g(\theta)) \\ &= 2 \cos(8m+1)\theta + \cos(8m+2)\theta + \cos(8m+4)\theta. \end{aligned} \quad (6.17)$$

We make the change of variables $\theta = \frac{3\pi}{8} + \alpha$ in order to express Q_R in terms of the small variable α . The individual cosine terms are

$$\begin{aligned} \cos(8m+1)\theta &= (-1)^m \cos\left(\frac{3\pi}{8} + (8m+1)\alpha\right) \\ &= (-1)^m \left(\sin \frac{\pi}{8} \cos(8m+1)\alpha - \cos \frac{\pi}{8} \sin(8m+1)\alpha\right), \end{aligned}$$

$$\begin{aligned} \cos(8m+2)\theta &= (-1)^m \cos\left(\frac{3\pi}{4} + (8m+2)\alpha\right) \\ &= (-1)^m \left(-\frac{\sqrt{2}}{2} \cos(8m+2)\alpha - \frac{\sqrt{2}}{2} \sin(8m+2)\alpha\right), \end{aligned}$$

$$\cos(8m+4)\theta = (-1)^m \sin(8m+4)\alpha.$$

In terms of α and removing the extraneous sign factor, the expression (6.17) for Q_R becomes

$$\begin{aligned} Q_R(\alpha) &= 2 \sin \frac{\pi}{8} \cos(8m+1)\alpha - 2 \cos \frac{\pi}{8} \sin(8m+1)\alpha - \frac{\sqrt{2}}{2} \cos(8m+2)\alpha \\ &\quad - \frac{\sqrt{2}}{2} \sin(8m+2)\alpha + \sin(8m+4)\alpha, \end{aligned} \quad (6.18)$$

and we intend to estimate the first positive zero.

Because the interval of admissibility is located very close to $\theta = \frac{3\pi}{8}$, linear approximations for the sine and cosine terms appearing in (6.18) should suffice to give an adequate estimate for the end-point of the interval. Even the full range $\left[\frac{3\pi}{8}, \frac{6m-1}{16m-3}\pi\right]$ is so short that none of the functions ϕ , ψ and g depart appreciably from their straight line approximations; this is evident from the plots of these functions in Figures 6.11 and 6.13. On this basis we use the approximations

$$\sin \alpha \approx \alpha \quad \text{and} \quad \cos \alpha \approx 1,$$

to obtain the linear approximation P_R to Q_R , which is

$$P_R(\alpha) = \left\{ -2 \cos \frac{\pi}{8} \cdot (8m+1) - \frac{\sqrt{2}}{2}(8m+2) + (8m+4) \right\} \alpha + 2 \sin \frac{\pi}{8} - \frac{\sqrt{2}}{2}. \quad (6.19)$$

The linear $P_R(\alpha)$ has a single zero, which we denote by α_R , and

$$\alpha_R = \frac{4 \sin \frac{\pi}{8} - \sqrt{2}}{(8\sqrt{2} + 32 \cos \frac{\pi}{8} - 16)m + (2\sqrt{2} + 4 \cos \frac{\pi}{8} - 8)}. \quad (6.20)$$

This is our estimate for the first zero of Q_R , and hence for the right end-point of the interval of admissibility.

The left end-point of the interval of admissibility, likewise the first intersection of the graphs of ϕ and g to the right of $\theta = \frac{3\pi}{8}$ may be estimated in the same manner. Setting $Q_L(\theta) = 2(\phi(\theta) - g(\theta))$, we find

$$Q_L(\theta) = 2 \cos(8m-7)\theta + 2 \cos(8m-4)\theta - \cos(8m+2)\theta + \cos(8m+4)\theta, \quad (6.21)$$

and with the aid of the additional cosine expressions

$$\begin{aligned} \cos(8m-7)\theta &= (-1)^m \cos\left(-\frac{21\pi}{8} + (8m-7)\alpha\right) \\ &= (-1)^m \left(\cos \frac{\pi}{8} \sin(8m-7)\alpha - \sin \frac{\pi}{8} \cos(8m-7)\alpha\right) \\ \cos(8m-4)\theta &= (-1)^{m+1} \sin(8m-4)\alpha, \end{aligned}$$

we can write Q_L in terms of the variable α :

$$\begin{aligned} Q_L(\alpha) &= 2 \cos \frac{\pi}{8} \sin(8m-7)\alpha - 2 \sin \frac{\pi}{8} \cos(8m-7)\alpha - 2 \sin(8m-4)\alpha \\ &\quad + \frac{\sqrt{2}}{2} \cos(8m+2)\alpha + \frac{\sqrt{2}}{2} \sin(8m+2)\alpha + \sin(8m+4)\alpha. \end{aligned} \quad (6.22)$$

Using $\sin \alpha \approx \alpha$ and $\cos \alpha \approx 1$, as above, gives the linear approximation

$$P_L(\alpha) = \left\{ 2 \cos \frac{\pi}{8} \cdot (8m-7) - 2(8m-4) + \frac{\sqrt{2}}{2}(8m+2) + (8m+4) \right\} \alpha - 2 \sin \frac{\pi}{8} + \frac{\sqrt{2}}{2} \quad (6.23)$$

and then the estimate for the left end-point of the interval of admissibility is

$$\alpha_L = \frac{4 \sin \frac{\pi}{8} - \sqrt{2}}{(8\sqrt{2} + 32 \cos \frac{\pi}{8} - 16)m + (2\sqrt{2} - 28 \cos \frac{\pi}{8} + 24)}. \quad (6.24)$$

Remarks on the interval end-point estimates

We close this section with some remarks about the estimates α_L and α_R , and the corresponding conclusions about the location and size of the interval of admissibility.

1. The estimates α_L and α_R for the interval end-points are in good numerical agreement with figures produced as a result of the computer search. Calculations, from the two sources, of the intervals of admissibility for the first five periodic sequences in the family may be compared in the following table :

| m | Interval from Computer Search | Interval Estimated as (α_L, α_R) |
|-----|-------------------------------|--|
| 1 | (1.182667, 1.183086) | (1.182607, 1.183076) |
| 2 | (1.180413, 1.180512) | (1.180395, 1.180511) |
| 3 | (1.179647, 1.179690) | (1.179639, 1.179690) |
| 4 | (1.179262, 1.179286) | (1.179257, 1.179286) |
| 5 | (1.179030, 1.179045) | (1.179027, 1.179045) |

2. The expressions for α_L and α_R take the form

$$\alpha_L = \frac{A}{Bm + C_1} \quad \text{and} \quad \alpha_R = \frac{A}{Bm + C_2}$$

where $C_1 > C_2$. (In particular, the formulae for α_L and α_R are consistent, in that $\alpha_L < \alpha_R$ for all m .) It follows that

$$\alpha_R - \alpha_L = \frac{A(C_2 - C_1)}{(Bm + C_1)(Bm + C_2)},$$

so that the length of the interval decreases in a reciprocal quadratic fashion with m (and consequently with the string length N). It is this, as well as the very short length of the first interval for the family, that accounts for the fact that all the intervals of

admissibility are so tiny, and that makes the computer search for the intervals such a challenging task.

3. Both α_L and α_R tend to 0 as $m \rightarrow \infty$, so the interval of admissibility approaches $\frac{3\pi}{8}$, as previously observed. But its distance to the right of $\frac{3\pi}{8}$ declines as $\frac{1}{m}$, compared to the $\frac{1}{m^2}$ decrease in its length.
4. If desired, the estimates α_L and α_R may be written wholly as algebraic quantities, eliminating the trigonometric terms by means of

$$\cos \frac{\pi}{8} = \frac{1}{2}(2 + \sqrt{2})^{\frac{1}{2}} \quad \text{and} \quad \sin \frac{\pi}{8} = \frac{1}{2}(2 - \sqrt{2})^{\frac{1}{2}}.$$

6.5 Description of the families of intervals of admissibility

The family of sequences which formed the subject of the analytical investigation in Section 6.4 was just one of several such families which could have been chosen. Reference to the plot of intervals of admissibility, Figure 6.8, shows that there are several groupings of intervals each of which might equally correspond to the start of an infinite family of admissible periodic sequences of the sort we have just isolated. Each stands out in the figure as an arrangement of points occurring at regularly spaced string lengths, each point representing an interval of admissibility. These points all lie on a smooth curve becoming flatter with increasing string length, eventually leveling indicating a limit point for the intervals of that family.

A careful study of this figure and the listing of intervals obtained from the computer search uncovered eight potential families, including the one we introduced and studied in the previous section. These families are highlighted in Figure 6.14, where all the intervals belonging to a family have been linked by a smooth curve. The properties describing these eight infinite families, which we name A1 to A8, are listed in Table 6.4.

The entries in the first row of Table 6.4, those corresponding to the family A1, are known by virtue of the mathematical treatment of that family included in the last section;

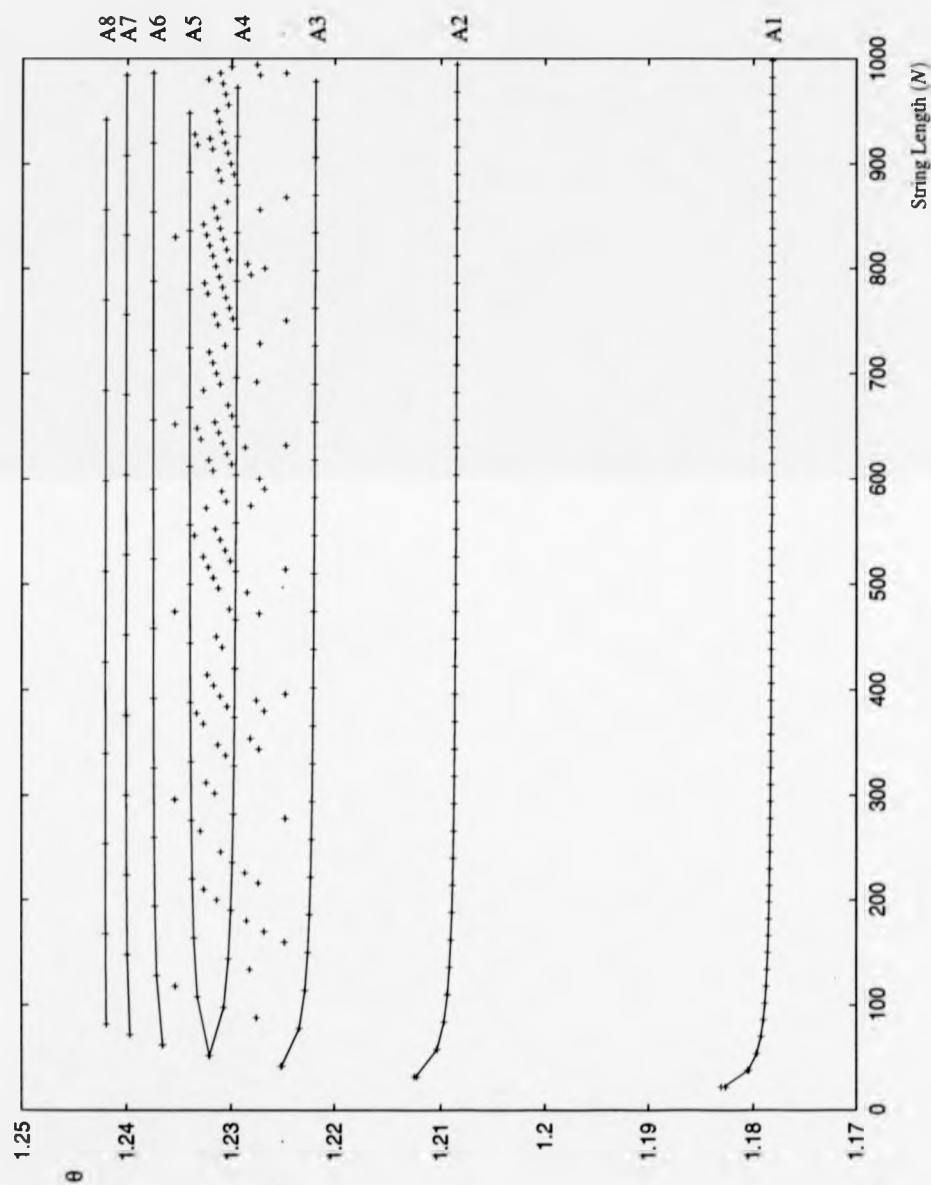


Figure 6.14: Plot of the intervals of admissibility for periodic sequences generated by strings $+00+0 \dots 0$, with the families A1 to A8 identified.

| Family | String Length (N) | Function Determining Left End-point | Trend in Location | Limiting Location |
|--------|--------------------------|--|-------------------|----------------------|
| A1 | $16m + 6$ | ϕ_{8m-6} | decreasing | $\frac{3\pi}{8}$ |
| A2 | $26m + 6$ | ϕ_{13m-11} | decreasing | $\frac{5\pi}{13}$ |
| A3 | $36m + 6$ | ϕ_{18m-16} | decreasing | $\frac{7\pi}{18}$ |
| A4 | $46m + 6$ | ϕ_{23m-21} | decreasing | $\frac{9\pi}{23}$ |
| A5 | $56m - 4$ | $\phi_{28m-31}^{(*)}$ | increasing | $\frac{11\pi}{28}$ |
| A6 | $66m - 4$ | $\phi_{33m-36}^{(*)}$ | increasing | $\frac{13\pi}{33}$ |
| A7 | $76m - 4$ | $\phi_{38m-41}^{(*)}$ | increasing | $\frac{15\pi}{38}$ |
| A8 | $86m - 4$ | $\phi_{43m-46}^{(*)}$ | increasing | $\frac{17\pi}{43}$ |

(*) These formulae for the function ϕ , determining the left end-point of the interval of admissibility give ϕ_{-3} when $m = 1$. In each instance the computer search reports that ϕ_2 is the relevant function. Substituting $r = -3$ in the formula for ϕ , produces a function identical to ϕ_2 : it is not then inappropriate to report the value $r = -3$ in this table.

Table 6.4: Descriptive properties of the families A1–A8.

the information in the rows for the families A2 to A8 has been inferred from data obtained from the computer search for intervals of admissibility. There is a consistent description here and we envisage that the techniques used to analyse family A1 could equally well be applied to the other families, justifying their existence as *infinite* families and confirming the characterisation given.

Table 6.4 also provides evidence that the families A1 to A8 fit together in a collective sense. We may identify certain arithmetic properties shared by these families :

- (i) The *increment* in string length between successive members of a single family *rises* by 10 as we move from one family to the next.
- (ii) The increment in the index of the function responsible for limiting the left end-point of the interval of admissibility, rises by 5 in the next family.
- (iii) The multiple of π , which is the limit of the intervals associated with a family, is a fraction whose numerator increases by 2 and whose denominator increases by 5, between succeeding families. Remarkably, this denominator coincides with one half of the increment in string length for its family.

Clearly the eight families identified so far do not account for all the intervals of admissibility of periodic sequences generated by strings $+00+0\dots 0$: in Figure 6.14 there are many intervals corresponding to plotted points not lying on any of the curves of the families A1 to A8. Even here there is some identifiable structure. If the families A1 to A8 are removed from the plot, and the vertical θ scale is magnified to enlarge the slice of the plot relevant for the intervals that now remain, a further set of latent families may be identified. In Figure 6.15 we highlight this second group of families within the plot of intervals that remain after the families A1 to A8 have been removed.

This process was applied repeatedly to our entire data set of intervals for periodic sequences generated by strings of length less than 1000, and the result was that several more groups of families could be isolated. Table 6.5 includes the descriptions of the first four groups. Note that these families have a similar characterisation to the original families A1 to A8, and that conditions (i), (ii) and (iii) above also hold for the new families B1 to B7, C1 to C7, and D1 to D9.

The spread of the limiting values of the locations of the intervals diminishes passing from one group to the next. For instance, the two extreme limiting values for the A group are $\frac{3\pi}{8} \approx 1.1781$ and $\frac{17\pi}{43} \approx 1.2420$, whereas the D group values are more tightly packed, $\frac{59\pi}{151} \approx 1.2275$ and $\frac{75\pi}{191} \approx 1.2336$. Likewise, the locations of intervals belonging to families from the later groups interlace the locations of intervals belonging to families from earlier groups. The overall effect is that as more and more groups of families are brought into the picture, increasingly they cluster together around a value near $\theta = 1.231$. This interlacing and clustering is illustrated in Figure 6.16. That figure shows the families within the four groups A, B, C and D highlighted against the plot of all the computer determined intervals, when the range for θ has been narrowed to the region where the families gather together.

Beyond these first four groups of families we begin to run out of data, and we would require information on intervals of admissibility for sequences of period greater than 1000 in order to write down further groups of families with confidence. Nevertheless, within the data we do have, there is certainly evidence for the existence of E, F and maybe subsequent

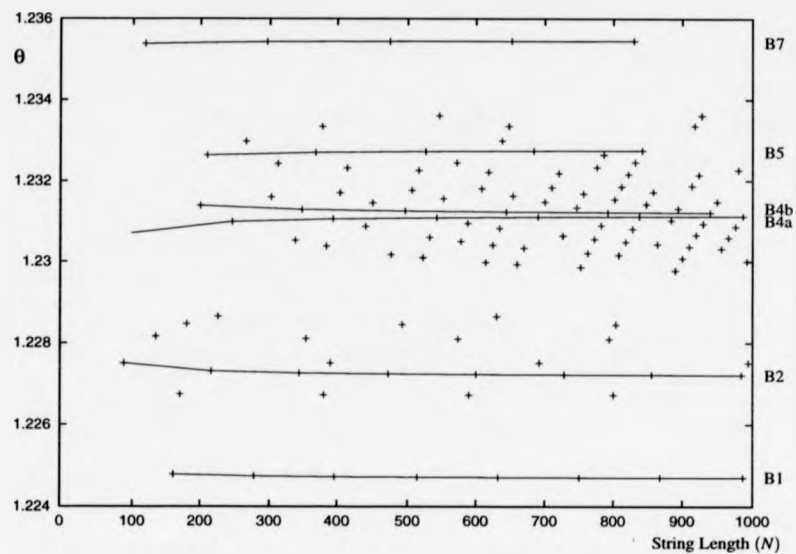


Figure 6.15: Plot of intervals of admissibility. The families A1 to A8 have been removed. A second group of families, B1 to B7, is identified.

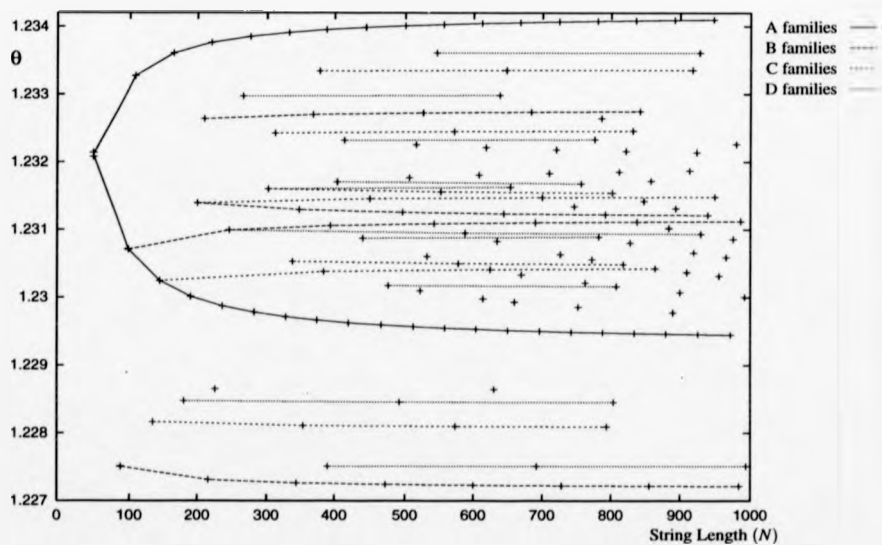


Figure 6.16: The vertical range for θ is narrowed, showing in greater detail how the families A1 to A8, B1 to B7, C1 to C7, and D1 to D9 interleave.

groups, suggesting that there is no limit to the number of these groups. However, at the time of writing, this must remain conjecture. Though there is evidence to suggest some arithmetical relationships between our various groups, we do not have a unified theory describing the entire scheme.

| Family | String Length (N) | Function Determining Left End-point | Trend in Location | Limiting Location |
|-------------------|--------------------------|--|-------------------|---------------------------------------|
| A1 | $16m + 6$ | Φ_{8m-6} | decreasing | $\frac{3\pi}{8}$ |
| A2 | $26m + 6$ | Φ_{13m-11} | decreasing | $\frac{5\pi}{13}$ |
| A3 | $36m + 6$ | Φ_{18m-16} | decreasing | $\frac{7\pi}{18}$ |
| A4 | $46m + 6$ | Φ_{23m-21} | decreasing | $\frac{9\pi}{23}$ |
| A5 | $56m - 4$ | Φ_{28m-31} | increasing | $\frac{11\pi}{28}$ |
| A6 | $66m - 4$ | Φ_{33m-36} | increasing | $\frac{13\pi}{33}$ |
| A7 | $76m - 4$ | Φ_{38m-41} | increasing | $\frac{15\pi}{38}$ |
| A8 | $86m - 4$ | Φ_{43m-46} | increasing | $\frac{17\pi}{43}$ |
| B1 | $118m + 42$ | Φ_{59m-39} | decreasing | $\frac{23\pi}{59}$ |
| B2 | $128m - 40$ | Φ_{64m-44} | decreasing | $\frac{25\pi}{64}$ |
| B3 ^(†) | $138m + ?$ | $\Phi_{69m+?}$ | decreasing | $\frac{27\pi}{69} = \frac{9\pi}{23}$ |
| B4a | $148m - 50$ | $\Phi_{74m-100}$ | increasing | $\frac{29\pi}{74}$ |
| B4b | $148m + 52$ | Φ_{74m-49} | decreasing | $\frac{29\pi}{74}$ |
| B5 | $158m + 52$ | Φ_{79m-54} | increasing | $\frac{31\pi}{79}$ |
| B6 ^(†) | $168m + ?$ | $\Phi_{84m+?}$ | increasing | $\frac{33\pi}{84} = \frac{11\pi}{28}$ |
| B7 | $178m - 60$ | Φ_{89m-59} | increasing | $\frac{35\pi}{89}$ |
| C1 | $210m - 40$ | $\Phi_{105m-126}$ | decreasing | $\frac{41\pi}{105}$ |
| C2 | $220m - 86$ | $\Phi_{110m-154}$ | decreasing | $\frac{43\pi}{110}$ |
| C3 ^(†) | $230m + ?$ | $\Phi_{115m+?}$ | decreasing | $\frac{45\pi}{115} = \frac{9\pi}{23}$ |
| C4a | $240m - 96$ | $\Phi_{120m-169}$ | increasing | $\frac{47\pi}{120}$ |
| C4b | $240m + 98$ | $\Phi_{120m-72}$ | decreasing | $\frac{47\pi}{120}$ |
| C5a | $250m - 50$ | $\Phi_{125m-151}$ | increasing | $\frac{49\pi}{125}$ |
| C5b | $250m + 52$ | $\Phi_{125m-100}$ | decreasing | $\frac{49\pi}{125}$ |
| C6 | $260m + 52$ | $\Phi_{130m-105}$ | increasing | $\frac{51\pi}{130}$ |
| C7 | $270m + 108$ | $\Phi_{135m-82}$ | increasing | $\frac{53\pi}{135}$ |
| D1 | $302m + 88$ | $\Phi_{151m-108}$ | decreasing | $\frac{59\pi}{151}$ |
| D2 | $312m - 132$ | $\Phi_{156m-223}$ | decreasing | $\frac{61\pi}{156}$ |
| D3 ^(†) | $322m + ?$ | $\Phi_{161m+?}$ | decreasing | $\frac{63\pi}{161} = \frac{9\pi}{23}$ |
| D4 | $332m + 144$ | $\Phi_{166m-95}$ | decreasing | $\frac{65\pi}{166}$ |

table continued overleaf ...

| Family | String Length (N) | Function Determining Left End-point | Trend in Location | Limiting Location |
|--------|--------------------------|--|-------------------|----------------------|
| D5a | $342m + 98$ | $\phi_{171m-123}$ | increasing | $\frac{67\pi}{171}$ |
| D5b | $342m - 96$ | $\phi_{171m-220}$ | decreasing | $\frac{67\pi}{171}$ |
| D6a | $352m - 50$ | $\phi_{176m-202}$ | increasing | $\frac{69\pi}{176}$ |
| D6b | $352m + 52$ | $\phi_{176m-151}$ | decreasing | $\frac{69\pi}{176}$ |
| D7 | $362m + 52$ | $\phi_{181m-156}$ | increasing | $\frac{71\pi}{181}$ |
| D8 | $372m - 106$ | $\phi_{186m-240}$ | increasing | $\frac{73\pi}{186}$ |
| D9 | $382m + 164$ | $\phi_{191m-110}$ | increasing | $\frac{75\pi}{191}$ |

(†) Some of the families duplicate ones which occur in previous groups. (For each the intervals coincide with a subset of the intervals belonging to an earlier family; there is not a clear choice for this subset, and a unique specification for the string lengths or the determining functions cannot be given.)

Table 6.5: Descriptions of the first four groups of families identifiable amongst the intervals of admissibility associated with the strings $+00+0\dots 0$.

6.6 An example of a periodic sequence which has two disjoint intervals of admissibility

Prior to the investigation of the generating strings $+00+0\dots 0$, all the available evidence, both computational and analytical, suggested that there is a single interval of θ values where a given periodic sequence is admissible. Examining the admissibility of the periodic sequences arising from the type of string considered here suggested that it might be possible to isolate an instance of a periodic sequence for which the range of admissibility is not connected, so not an interval.

This indeed proved to be the case : the string $+00+0\dots 0$ of length $N = 118$ provides an example, as we show. If we take $\theta_1 = 1.178768$, $\theta_2 = 1.2$, and $\theta_3 = 1.235382$, then

$$|\psi(\theta_2)| = 1.63976092 \quad |g(\theta_2)| = 0.9259717147$$

so the associated periodic sequence *is not* admissible throughout a neighbourhood of θ_2 ,

whilst

$$\max_{0 \leq r \leq N-1} |f_r(\theta_1)| = 0.3869892665 \quad |g(\theta_1)| = 0.3871541831$$

and

$$\max_{0 \leq r \leq N-1} |f_r(\theta_3)| = 0.5569749016 \quad |g(\theta_3)| = 0.5570058701$$

so correspondingly the associated periodic sequence is admissible throughout some neighbourhood of each of θ_1 and θ_3 . We now describe how examples like this one may be anticipated, and produced.

In Section 6.5 we identified several infinite families of admissible periodic sequences generated by strings $+00+0\dots 0$, and were able to provide formulae specifying the string lengths for the members of each of the several families. There are cases where two or more of these formulae generate the same string length, suggesting that there may be some periodic sequences which belong to more than one family. This being so, we would expect any such sequence to be admissible within at least two disjoint intervals of θ values. (The reason is that, as we have seen above, the intervals associated with two different families keep well away from one another, because their limiting locations are different and the interval lengths decrease faster than the rate at which the intervals change their location. One might suppose that the ideas from Section 6.4 could be used to isolate the intervals from two different families in two non-overlapping ranges, certainly for large enough string lengths.)

Each formula describing the length N of any string generating a sequence from one of the families identified in the previous section is of the form $N = em + c$, where the integers e and c are specific to the particular family and $m = 1, 2, 3, \dots$. Effectively then the formula is the congruence $N \equiv c \pmod{e}$. For the group of families A1 to A8, these congruences are :

| Family | Congruence Governing String Lengths |
|--------|--|
| A1 | $N \equiv 6 \ (16)$ |
| A2 | $N \equiv 6 \ (26)$ |
| A3 | $N \equiv 6 \ (36)$ |
| A4 | $N \equiv 6 \ (46)$ |
| A5 | $N \equiv -4 \ (56)$ |
| A6 | $N \equiv -4 \ (66)$ |
| A7 | $N \equiv -4 \ (76)$ |
| A8 | $N \equiv -4 \ (86)$ |

Producing strings common to several families is thus the task of solving a set of simultaneous congruences. The standard tool to do this is the Chinese Remainder Theorem.

Chinese Remainder Theorem

The solutions of the simultaneous congruences $x \equiv c_i \ (e_i)$, when the moduli e_i are coprime, coincide with solutions of the single congruence $x \equiv c \ (E)$, where $E = \prod_i e_i$, $E_i = \frac{E}{e_i}$, r_i is any solution of $E_i r_i \equiv 1 \ (e_i)$, and $c = \sum_i c_i r_i E_i$.

Remark : For the special case when all the c_i 's are equal, which is so with several of our examples below, there is a simpler statement of the theorem in which the single equivalent congruence shares the same right-hand side common to the given simultaneous congruences. The reason is that when each $c_i = a$, then $c = a \sum_i r_i E_i$; but $\sum_i r_i E_i \equiv 1 \ (E)$ because $\sum_i r_i E_i - 1$ is divisible by each of the moduli and so by their product, hence $c \equiv a \ (E)$.

In our calculations the moduli are not coprime in pairs, so the Chinese Remainder Theorem may not be applied directly. We require the following additional result.

Lemma 6.4

The congruences $x \equiv a \ (p^{\alpha_1})$ and $x \equiv b \ (p^{\alpha_2})$ involving a prime p and exponents $\alpha_1 > \alpha_2$, have a common solution if and only if $a \equiv b \ (p^{\alpha_2})$. In this case the solutions are simply the solutions of the first congruence.

Proof

If $x \equiv a \ (p^{\alpha_1})$ then $x \equiv a \ (p^{\alpha_2})$; so $x \equiv b \ (p^{\alpha_2})$ precisely when $a \equiv b \ (p^{\alpha_2})$. □

This lemma can be used in conjunction with the Chinese Remainder Theorem to reduce any simultaneous system of congruences (consistent, in that there is a solution) to a system with coprime moduli. A further application of the Chinese Remainder Theorem will then produce the solution via a single equivalent congruence.

We now provide a series of examples to illustrate the calculation of string lengths for which the corresponding periodic sequences have several intervals of admissibility.

Example 1. Strings common to the families A1 and A2

The lengths N of strings which occur in both families A1 and A2 satisfy the congruences

$$\begin{cases} N \equiv 6 & (16) \\ N \equiv 6 & (26) \end{cases} \quad \text{i.e.} \quad \begin{cases} N \equiv 6 & (2^4) \\ N \equiv 6 & (2 \cdot 13) \end{cases} \quad (6.25)$$

By the Chinese Remainder Theorem the single congruence $N \equiv 6 \pmod{26}$ is equivalent to the two congruences $N \equiv 6 \pmod{13}$ and $N \equiv 6 \pmod{2}$. And by Lemma 6.4, the congruences $N \equiv 6 \pmod{2^4}$ and $N \equiv 6 \pmod{2}$ are simultaneously soluble, and equivalent to $N \equiv 6 \pmod{2^4}$. Thus the original two congruences are equivalent to the pair

$$\begin{cases} N \equiv 6 & (2^4) \\ N \equiv 6 & (13), \end{cases}$$

now with coprime moduli. Their solution follows from the Chinese Remainder Theorem through $N \equiv 6 \pmod{2^4 \cdot 13}$, i.e. $N \equiv 6 \pmod{208}$.

Hence the lengths N for which a string of the form $+00+0\dots 0$ is a member of both family A1 and family A2 are $N = 214, 422, 630, 838, 1046, \dots$. The results of this calculation may be cross-checked with the results of the computer search, and at each of these lengths we do find an interval from each of the two families, and these intervals are disjoint. The relevant entries collated from the table reporting the results of our computer search are listed below. (At lengths $N = 630$ and 838 , there are in fact 3 intervals of admissibility, where the third interval belongs to another family.)

| String Length (N) | Interval of Admissibility (Computer Determined) | Function Determining Left End-point | Family to which Interval Belongs |
|--------------------------|--|--|-------------------------------------|
| 214 | (1.178456845, 1.178459065) | ϕ_{98} | A1 |
| | (1.208811195, 1.208813972) | ϕ_{93} | A2 |
| 422 | (1.178277175, 1.178277728) | ϕ_{202} | A1 |
| | (1.20855811, 1.208558792) | ϕ_{197} | A2 |
| 630 | (1.178217227, 1.178217472) | ϕ_{306} | A1 |
| | (1.20847371, 1.208474011) | ϕ_{301} | A2 |
| | (1.228645306, 1.228645314) | ϕ_{112} | E |
| 838 | (1.178187242, 1.17818738) | ϕ_{410} | A1 |
| | (1.208431504, 1.208431673) | ϕ_{405} | A2 |
| | (1.231115068, 1.231115083) | ϕ_{344} | B4a |

Example 2. Choice of families for the most frequently recurring pairings

Selecting strings common to families A1 and A2 gives us a series of lengths for which there are (at least) two disjoint intervals of admissibility, and we found above that the increment in length between two successive strings with this property is 208. Repeating the calculation for other pairs of families gives different series of lengths where we may expect two disjoint intervals, and we can find a pair of families for which the increment in length between successive strings belonging to both families is shorter than it is for the families A1 and A2.

For the shortest possible increment we need to select the two families so that the corresponding congruences are simultaneously soluble and which provide moduli whose least common multiple is as small as possible. The optimal choice is the pair A1 and A3, where the congruences are $N \equiv 6 \pmod{2^4}$ and $N \equiv 6 \pmod{2^2 \cdot 3^2}$, and so the solutions satisfy $N \equiv 6 \pmod{144}$, i.e. string lengths $N = 150, 294, 438, \dots$. It should also be noted that a consequence of Lemma 6.4 is that not every pair of families have strings in common; for example, the congruences for the families A1 and A5 have no shared solution.

Before we move on to consider instances where more than two intervals of admissibility are present, we will briefly return to the example given at the beginning of this section involving the string of length $N = 118$. The two intervals of admissibility arise because the

string length satisfies the congruences for two different families. Referring to Tables 6.2 and 6.5 we see that these families are A1 and B7. We chose this length as our introductory example since, as the results from the computer search show, the string $+00+0 \dots 0$ of length $N = 118$ is the shortest giving rise to two intervals of admissibility.

Example 3. Strings with three intervals of admissibility

To give an example where there are three disjoint intervals of admissibility we seek strings common to the families A2, A4 and A7. These families have been selected because the first string common to all three families is short enough to appear amongst the results of our computer search. The length N of the string must simultaneously satisfy the congruences

$$\begin{cases} N \equiv 6 & (2, 13) \\ N \equiv 6 & (2, 23) \\ N \equiv -4 & (2^2, 19). \end{cases} \quad (6.26)$$

By the Chinese Remainder Theorem these are equivalent to the set of congruences

$$\begin{cases} N \equiv 6 & (13) \\ N \equiv 6 & (23) \\ N \equiv 6 & (2) \\ N \equiv -4 & (2^2) \\ N \equiv -4 & (19), \end{cases}$$

and Lemma 6.4 says that $N \equiv 6 \pmod{2}$ and $N \equiv -4 \pmod{2^2}$ are equivalent to $N \equiv 0 \pmod{2^2}$. Thus a set equivalent to the original congruences, but with coprime moduli is

$$\begin{cases} N \equiv 6 & (13) \\ N \equiv 6 & (23) \\ N \equiv 0 & (2^2) \\ N \equiv -4 & (19). \end{cases}$$

To find a solution by the Chinese Remainder Theorem we first must solve for the r_i 's referred to in its statement. We form $E = e_1 e_2 e_3 e_4 = 22724$, $E_1 = 1748$, $E_2 = 988$,

$E_3 = 5681$ and $E_4 = 1196$ and the r_i 's are obtained from the following congruences :

$$1748 r_1 \equiv 1 \quad (13) \quad \text{i.e. } r_1 = -2$$

$$988 r_2 \equiv 1 \quad (23) \quad \text{i.e. } r_2 = -1$$

$$1196 r_4 \equiv 1 \quad (19) \quad \text{i.e. } r_4 = -1$$

so $c = \sum_i r_i E_i c_i = -22120$. Thus the solution is $N \equiv -22120 \pmod{22724}$, i.e. $N \equiv 604 \pmod{22724}$. Reference to the table of results from the computer search shows that there are three intervals when the string length $N = 604$:

| String Length (N) | Interval of Admissibility (Computer Determined) | Function Determining Left End-point | Family to which Interval Belongs |
|--------------------------|--|--|-------------------------------------|
| 604 | (1.20848105, 1.208481378) | ϕ_{288} | A2 |
| | (1.229531091, 1.229531169) | ϕ_{278} | A4 |
| | (1.24005548, 1.240055604) | ϕ_{263} | A7 |

Note the sparsity with which members of these three families come together : after $N = 604$, the next string shared by A2, A4 and A7 has length 23328. The most frequently occurring instances where there is a triple of intervals will derive from the families A1, A2 and A3, and exist at lengths $N \equiv 6 \pmod{1872}$, so the first such length is $N = 1878$, and subsequent instances occur at steps of 1872.

Example 4. Strings having many intervals of admissibility

The techniques used in the first three examples can clearly be generalised to find lengths at which there are strings with more than three disjoint intervals. For instance, the most frequently occurring case of four intervals will derive from strings common to A1, A2, A3 and A4 with solution $N \equiv 6 \pmod{43056}$. (This, of course, takes us way beyond lengths which could easily be verified by computer, because of the tiny interval sizes and the large number of function evaluations needed.)

It is plausible that the approach could be extended to give lengths at which any given number of intervals of admissibility coexist, but we cannot say this with certainty because we lack a complete description of all the groups of families. To have, say, n coexisting

intervals would require n families whose congruences had a mutual solution; this could be ensured if, for example, there were n families whose moduli were twice a prime. We might mention in this connection Dirichlet's theorem, which states that an arithmetic progression of the form $am + b$, $m = 1, 2, 3, \dots$ contains an infinity of primes provided a and b are coprime. Thus if we could find such an arithmetic progression with $2(am + b)$ amongst the moduli of the families, then the result would be confirmed. Looking at the arithmetic properties met with in our first four collections of families, it is not inconceivable for such a progression to exist.

Chapter Summary

- Strings having the pattern $+00+0\dots 0$ are examined as potential generators of admissible periodic sequences. The situation found differs in two respects from that for the families studied in Chapter 5; namely, intervals of admissibility are more prolific but each individual interval is typically much shorter in length than those previously encountered.
- The admissible periodic sequences so obtained do not constitute a single family; rather the trends in the locations of their intervals of admissibility indicate a complex collection of families.
- The extremely small extent of each interval of admissibility means that it is not viable to employ computational searches based on regularly spaced parameter values in order to find instances of admissible periodic sequences generated by the strings $+00+0\dots 0$. Instead an adaptive search strategy has permitted the compilation of a table of intervals of admissibility for generating string length N up to 1000.
- For each string $+00+0\dots 0$ of length $N = 16m + 6$, $m = 1, 2, 3, \dots$, an interval of θ values is specified throughout which the periodic sequence generated by that string is admissible. Approximation techniques are used to show that the sequence of intervals converges to the limiting value $\theta = \frac{3\pi}{8}$. The analytical tools employed may be expected to permit a similar investigation, with similar conclusions, for other of the infinite families of periodic sequences identified that share the pattern $+00+0\dots 0$.
- The string $+00+0\dots 0$ of length $N = 118$ generates a periodic sequence having two disjoint intervals of admissibility; we believe this to be the first reported example of such a sequence. Other examples of sequences with two or more disjoint intervals of admissibility can be found by solving arithmetic congruences in order to find a length N where the generating string is simultaneously a member of several of our families.

Chapter 7

Periodic Sequences Admissible in an Interval Having Zero as Left End-point : Classification of Odd Length Generating Strings

A striking feature of the distribution of the admissible periodic sequences over the parameter range is their concentration close to zero. Inspection of our computer-generated lists of the sequences reveals that this is in large measure accounted for by the presence of some substantial coherent collection of sequences. Attributes that distinguish the sequences of this collection from others appearing in a more sporadic fashion are :

- *The sequences occur at consecutive string lengths, from 1 to some maximum length which declines as the parameter θ increases; in particular the closer to zero one looks, the greater the proportion present in the lists.*
- *The interval of admissibility for each of these sequences extends, on the left, as far as zero.*
- *For each string length N , all the intervals of admissibility are identical, with right end-point $\frac{\pi}{N}$.*

The regularity inherent in these observations stands out when the lists are surveyed over the complete parameter range.

Taking as the starting point of a theoretical investigation the property that a periodic sequence has interval of admissibility extending on the left all the way to zero, we systematically identify all the sequences so characterised, obtaining a classification that encompasses, for the appropriate string lengths, exactly the concentration of sequences close to zero within the computer-generated lists that we have remarked upon. A differential form of the Chua-Lin inequalities is a key tool for the analysis. For the moment our development is concerned only with strings of odd length, and we defer consideration of the even case to the next chapter.

7.1 The specialised form of the Chua-Lin inequalities at zero

As an introductory example consider the length $N = 7$ string $+-0+-00$, taken from the list of sequences for $\theta = 0.15$ in Table 1 of Volume II. This string generates a periodic sequence admissible in the interval $(0, \frac{\pi}{7})$. Plots of the functions f_0, f_1, \dots, f_6 and $\pm g$ appearing in the Chua-Lin inequalities are shown in Figure 7.1, and in accordance with these inequalities the envelope of $\pm g$ includes, over the range $(0, \frac{\pi}{7})$, all the f_r .

In Section 2.1.4 we have seen that if a string $s_0 s_1 \dots s_{N-1}$ generates a periodic sequence admissible in some interval then the Chua-Lin inequalities

$$|f_r(\theta)| < |g(\theta)| \quad \text{for } r = 0, 1, \dots, N-1 \quad (7.1)$$

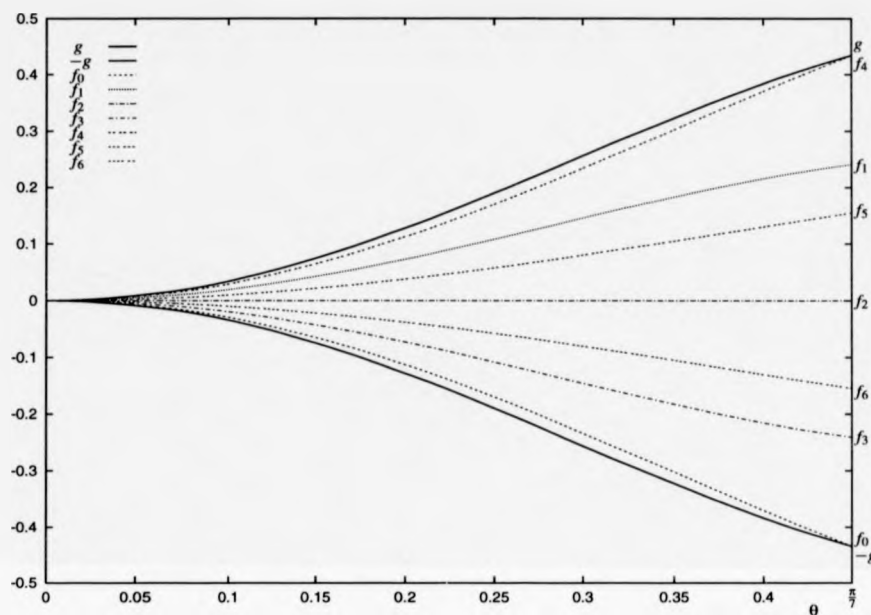
hold for each θ within the interval (and conversely), where

$$f_r(\theta) = \sum_{i=0}^{N-1} s_{i+r} \cos \frac{N-2i}{2} \theta \quad (7.2)$$

and

$$g(\theta) = \sin \frac{N}{2} \theta \sin \theta. \quad (7.3)$$

On specialising to the case of intervals *having zero as left end-point*, it follows that each

Figure 7.1: Plots of f_0, f_1, \dots, f_6 and $\pm g$ for the string $+ - 0 + - 0 0$.

$f_r(0) = 0$ because $g(0) = 0$, that is,

$$s_0 + s_1 + \dots + s_{N-1} = 0. \quad (7.4)$$

Noting that $f'_0(0) = 0$ and $g'(0) = 0$, we see that

$$g(\theta) - f_0(\theta) = \frac{1}{2} (g''(0) - f''_0(0)) \theta^2 + \frac{1}{6} (g'''(\eta) - f'''_0(\eta)) \theta^3 \quad (7.5)$$

for some η such that $0 < \eta < \theta$. Taking the limit as $\theta \rightarrow 0$, $f_0(\theta) < g(\theta)$ implies

$$f''_0(0) \leq g''(0).$$

Similarly $f''_0(0) \geq -g''(0)$. We thus obtain the inequality

$$-N \leq \frac{N^2}{4} s_0 + \frac{(N-2)^2}{4} s_1 + \dots + \frac{(N-2)^2}{4} s_{N-1} \leq N. \quad (7.6)$$

Using (7.4) to eliminate s_0 we find

$$\begin{aligned}
 \frac{N^2}{4}s_0 + \frac{(N-2)^2}{4}s_1 + \cdots + \frac{(N-2)^2}{4}s_{N-1} &= \frac{N^2}{4}s_0 + \sum_{i=1}^{\frac{1}{2}(N-1)} \frac{(N-2i)^2}{4}(s_i + s_{N-i}) \\
 &= \frac{N^2}{4} \left(- \sum_{i=1}^{\frac{1}{2}(N-1)} (s_i + s_{N-i}) \right) + \sum_{i=1}^{\frac{1}{2}(N-1)} \frac{(N-2i)^2}{4}(s_i + s_{N-i}) \\
 &= - \sum_{i=1}^{\frac{1}{2}(N-1)} i(N-i)(s_i + s_{N-i}), \tag{7.7}
 \end{aligned}$$

which is an even integer because in each term of this final sum either i or $(N-i)$ is even. It then follows that the inequality $-g''(0) \leq f_0''(0) \leq g''(0)$ can be strengthened to give

$$-(N-1) \leq \frac{N^2}{4}s_0 + \frac{(N-2)^2}{4}s_1 + \cdots + \frac{(N-2)^2}{4}s_{N-1} \leq (N-1). \tag{7.8}$$

The foregoing applies equally to the comparison of each $f_r''(0)$ to $g''(0)$.

In summary, for a string $s_0s_1 \dots s_{N-1}$ which generates a periodic sequence admissible in an interval having zero as left end-point, we have

$$s_0 + s_1 + \cdots + s_{N-1} = 0$$

$$\begin{aligned}
 -(N-1) &\leq \frac{N^2}{4}s_0 + \frac{(N-2)^2}{4}s_1 + \cdots + \frac{(N-2)^2}{4}s_{N-1} \leq (N-1) \\
 -(N-1) &\leq \frac{N^2}{4}s_1 + \frac{(N-2)^2}{4}s_2 + \cdots + \frac{(N-2)^2}{4}s_0 \leq (N-1) \\
 -(N-1) &\leq \frac{N^2}{4}s_2 + \frac{(N-2)^2}{4}s_3 + \cdots + \frac{(N-2)^2}{4}s_1 \leq (N-1) \\
 &\vdots \\
 -(N-1) &\leq \frac{N^2}{4}s_{N-1} + \frac{(N-2)^2}{4}s_0 + \cdots + \frac{(N-2)^2}{4}s_{N-2} \leq (N-1).
 \end{aligned} \tag{7.9}$$

It is equally true that a string $s_0s_1 \dots s_{N-1}$ which satisfies these inequalities generates a periodic sequence admissible in an interval having zero as left end-point. To justify this, note that each $f_r(0) = 0$ because of the condition $s_0 + s_1 + \cdots + s_{N-1} = 0$, that each

$f'_r(0) = 0$ because the sine function vanishes at 0, and consequently that

$$g(\theta) - f_r(\theta) = \frac{1}{2} (g''(\eta) - f''_r(\eta)) \theta^2 \quad (7.10)$$

for $0 < \eta < \theta$. But the inequality

$$N > (N-1) \geq \frac{N^2}{4} s_r + \frac{(N-2)^2}{4} s_{r+1} + \cdots + \frac{(N-2)^2}{4} s_{r-1} \quad (7.11)$$

implies that $g''(0) - f''_r(0) > 0$, and so by continuity $g''(\eta) - f''_r(\eta) > 0$ in an interval extending to the right of 0. Thus, in this same interval, $g(\theta) > f_r(\theta)$. By this means we can find an interval for θ having zero as left end-point throughout which $-g(\theta) < f_r(\theta) < g(\theta)$.

7.2 A transformed version of the basic inequalities

We established in the last section that the periodic sequences admissible in an interval having zero as left end-point are precisely those generated by a string $s_0 s_1 \dots s_{N-1}$ which satisfies

$$s_0 + s_1 + \cdots + s_{N-1} = 0$$

and

$$\begin{aligned} -(N-1) &\leq \frac{N^2}{4} s_0 + \frac{(N-2)^2}{4} s_1 + \cdots + \frac{(N-2)^2}{4} s_{N-1} \leq (N-1) \\ -(N-1) &\leq \frac{N^2}{4} s_1 + \frac{(N-2)^2}{4} s_2 + \cdots + \frac{(N-2)^2}{4} s_0 \leq (N-1) \\ -(N-1) &\leq \frac{N^2}{4} s_2 + \frac{(N-2)^2}{4} s_3 + \cdots + \frac{(N-2)^2}{4} s_1 \leq (N-1) \\ &\vdots \\ -(N-1) &\leq \frac{N^2}{4} s_{N-1} + \frac{(N-2)^2}{4} s_0 + \cdots + \frac{(N-2)^2}{4} s_{N-2} \leq (N-1). \end{aligned}$$

To motivate the next step, we consider the above in full when $N = 7$:

$$s_0 + s_1 + s_2 + s_3 + s_4 + s_5 + s_6 = 0$$

$$\begin{aligned}
-6 &\leq \frac{49}{4}s_0 + \frac{25}{4}s_1 + \frac{9}{4}s_2 + \frac{1}{4}s_3 + \frac{1}{4}s_4 + \frac{9}{4}s_5 + \frac{25}{4}s_6 \leq 6 \\
-6 &\leq \frac{25}{4}s_0 + \frac{49}{4}s_1 + \frac{25}{4}s_2 + \frac{9}{4}s_3 + \frac{1}{4}s_4 + \frac{1}{4}s_5 + \frac{9}{4}s_6 \leq 6 \\
-6 &\leq \frac{9}{4}s_0 + \frac{25}{4}s_1 + \frac{49}{4}s_2 + \frac{25}{4}s_3 + \frac{9}{4}s_4 + \frac{1}{4}s_5 + \frac{1}{4}s_6 \leq 6 \\
-6 &\leq \frac{1}{4}s_0 + \frac{9}{4}s_1 + \frac{25}{4}s_2 + \frac{49}{4}s_3 + \frac{25}{4}s_4 + \frac{9}{4}s_5 + \frac{1}{4}s_6 \leq 6 \\
-6 &\leq \frac{1}{4}s_0 + \frac{1}{4}s_1 + \frac{9}{4}s_2 + \frac{25}{4}s_3 + \frac{49}{4}s_4 + \frac{25}{4}s_5 + \frac{9}{4}s_6 \leq 6 \\
-6 &\leq \frac{9}{4}s_0 + \frac{1}{4}s_1 + \frac{1}{4}s_2 + \frac{9}{4}s_3 + \frac{25}{4}s_4 + \frac{49}{4}s_5 + \frac{25}{4}s_6 \leq 6 \\
-6 &\leq \frac{25}{4}s_0 + \frac{9}{4}s_1 + \frac{1}{4}s_2 + \frac{1}{4}s_3 + \frac{9}{4}s_4 + \frac{25}{4}s_5 + \frac{49}{4}s_6 \leq 6.
\end{aligned}$$

It is difficult to see how to extract the information contained within these conditions in their present form. The transformation scheme we now describe was developed in an attempt to help reveal their content. We set

$$\begin{aligned}
t_0 &= s_0, \quad t_1 = s_0 + s_1, \quad t_2 = s_0 + s_1 + s_2, \quad t_3 = s_0 + s_1 + s_2 + s_3, \quad t_4 = s_0 + s_1 + s_2 + s_3 + s_4, \\
t_5 &= s_0 + s_1 + s_2 + s_3 + s_4 + s_5, \quad t_6 = s_0 + s_1 + s_2 + s_3 + s_4 + s_5 + s_6 = 0
\end{aligned}$$

so that $s_0 = t_0$, $s_1 = t_1 - t_0$, $s_2 = t_2 - t_1$, $s_3 = t_3 - t_2$, $s_4 = t_4 - t_3$, $s_5 = t_5 - t_4$, $s_6 = -t_5$, and then the inequalities become :

$$\begin{aligned}
-3 &\leq 3t_0 + 2t_1 + t_2 - t_4 - 2t_5 \leq 3 \\
-3 &\leq -3t_0 + 3t_1 + 2t_2 + t_3 - t_5 \leq 3 \\
-3 &\leq -2t_0 - 3t_1 + 3t_2 + 2t_3 + t_4 \leq 3 \\
-3 &\leq -t_0 - 2t_1 - 3t_2 + 3t_3 + 2t_4 + t_5 \leq 3 \\
-3 &\leq -t_1 - 2t_2 - 3t_3 + 3t_4 + 2t_5 \leq 3 \\
-3 &\leq t_0 - t_2 - 2t_3 - 3t_4 - 3t_5 \leq 3 \\
-3 &\leq 2t_0 + t_1 - t_3 - 2t_4 - 3t_5 \leq 3.
\end{aligned}$$

A further change of variables to partial sums of the t_r 's via

$$\begin{aligned}
u_0 &= t_0, \quad u_1 = t_0 + t_1, \quad u_2 = t_0 + t_1 + t_2, \quad u_3 = t_0 + t_1 + t_2 + t_3, \quad u_4 = t_0 + t_1 + t_2 + t_3 + t_4, \\
u_5 &= t_0 + t_1 + t_2 + t_3 + t_4 + t_5
\end{aligned}$$

gives

$$\begin{aligned}
 -3 &\leq u_0 + u_1 + u_2 + u_3 + u_4 - 2u_5 && \leq 3 \\
 -3 &\leq u_0 + u_1 + u_2 + u_3 + u_4 - u_5 - 7u_0 && \leq 3 \\
 -3 &\leq u_0 + u_1 + u_2 + u_3 + u_4 && - 7u_1 \leq 3 \\
 -3 &\leq u_0 + u_1 + u_2 + u_3 + u_4 + u_5 - 7u_2 && \leq 3 \\
 -3 &\leq u_0 + u_1 + u_2 + u_3 + u_4 + 2u_5 - 7u_3 && \leq 3 \\
 -3 &\leq u_0 + u_1 + u_2 + u_3 + u_4 + 3u_5 - 7u_4 && \leq 3 \\
 -3 &\leq u_0 + u_1 + u_2 + u_3 + u_4 - 3u_5 && \leq 3,
 \end{aligned}$$

which are simpler than the original inequalities because the quadratic coefficients of the s_r 's have, for the most part, been converted into constant coefficients of the u_r 's, by means of the two partial summations.

The general transformation scheme

After working with a number of versions it was found that the most effective transformation to the u_r variables of the inequalities (7.9) governing the s_r 's is :

$$\begin{aligned}
 t_0 &= s_0, \quad t_1 = s_0 + s_1, \quad t_2 = s_0 + s_1 + s_2, \quad \dots, \quad t_{N-2} = s_0 + s_1 + \dots + s_{N-2}, \\
 t_{N-1} &= s_0 + s_1 + \dots + s_{N-1} \quad (\text{so } t_{N-1} = 0),
 \end{aligned} \tag{7.12}$$

$$\begin{aligned}
 u_0 &= t_0, \quad u_1 = t_0 + t_1, \quad u_2 = t_0 + t_1 + t_2, \quad \dots, \quad u_{N-2} = t_0 + t_1 + \dots + t_{N-2}, \\
 u_{N-1} &= t_0 + t_1 + \dots + t_{N-1} \quad (\text{so } u_{N-2} = u_{N-1}, \text{ because } t_{N-1} = 0).
 \end{aligned} \tag{7.13}$$

The N inequalities (7.9), in terms of the u_r 's, become

$$-\frac{1}{2}(N-1) \leq u_0 + u_1 + \dots + u_{N-1} - \left(\frac{1}{2}(N-1) - r\right) u_{N-1} - Nu_r \leq \frac{1}{2}(N-1)$$

for $r = 0, 1, \dots, N-1$. If further we let $x = u_{N-1}$ and $y = u_0 + u_1 + \dots + u_{N-1}$, then

$$-\frac{1}{2}(N-1) \leq y - \left(\frac{1}{2}(N-1) - r\right) x - Nu_r \leq \frac{1}{2}(N-1) \tag{7.14}$$

for $r = 0, 1, \dots, N-1$; note that here $u_r = x$ when $r = N-2$ or $N-1$.

7.3 Restrictions on the format of strings generating periodic sequences admissible adjacent to zero

In this section we prove two preparatory results allowing us to describe in broad terms the structure of the strings we are concerned with and which will provide the starting point of our detailed classification. To keep our statements concise it will help to abbreviate some of the terminology. In this chapter, and throughout Chapters 8 and 9, the phrase "admissible adjacent to zero" is used as a short way of expressing "admissible in an interval having zero as left end-point", and this comment is also intended to apply to the above heading.

Proposition 7.1

The substring $+0\dots 0+$ cannot appear within a string $s_0s_1\dots s_{N-1}$ generating a periodic sequence admissible adjacent to zero.

Proof

Case $r = N - 2$ of the inequalities (7.14) (in which $u_{N-2} = x$) is

$$-\frac{1}{2}(N-1) \leq y - \left(\frac{1}{2}(N-1) - (N-2)\right)x - Nx \leq \frac{1}{2}(N-1)$$

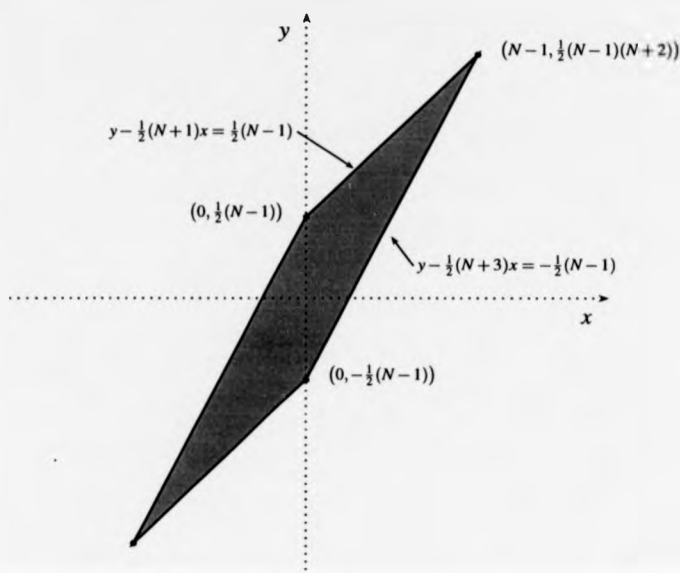
i.e.

$$-\frac{1}{2}(N-1) \leq y - \frac{1}{2}(N+3)x \leq \frac{1}{2}(N-1). \quad (7.15)$$

Likewise case $r = N - 1$ is

$$-\frac{1}{2}(N-1) \leq y - \frac{1}{2}(N+1)x \leq \frac{1}{2}(N-1). \quad (7.16)$$

In the xy -plane, inequalities (7.15) and (7.16) determine the parallelogram enclosed by the lines $y - \frac{1}{2}(N+3)x = \pm \frac{1}{2}(N-1)$ and $y - \frac{1}{2}(N+1)x = \pm \frac{1}{2}(N-1)$, shown shaded in Figure 7.2.

Figure 7.2: Region of xy -plane determined by inequalities (7.15) and (7.16)

From the remainder of the inequalities (7.14),

$$Nu_r \leq y - \left(\frac{1}{2}(N-1) - r\right)x + \frac{1}{2}(N-1) \quad (7.17)$$

for $r = 0, 1, \dots, N-3$. Each line $y - \left(\frac{1}{2}(N-1) - r\right)x = \text{constant}$ in the xy -plane has gradient $\leq \frac{1}{2}(N-1)$, which in turn is $< \frac{1}{2}(N+1)$, so when x and y are constrained as in (7.15) and (7.16) the maximum value of $y - \left(\frac{1}{2}(N-1) - r\right)x$ is

$$\frac{1}{2}(N-1)(N+2) - \left(\frac{1}{2}(N-1) - r\right)(N-1) = \frac{1}{2}(N-1)(2r+3). \quad (7.18)$$

Thus from (7.17)

$$Nu_r \leq \frac{1}{2}(N-1)(2r+3) + \frac{1}{2}(N-1) = (N-1)(r+2) \quad (7.19)$$

i.e.

$$u_r \leq (r+2)\frac{N-1}{N} < r+2 \quad (7.20)$$

and so $u_r \leq r+1$, because u_r is an integer.

If $+0\dots 0+$ appeared as a substring of $s_0s_1\dots s_{N-1}$, by cycling we could suppose that it appears at the start of the string. But then the initial t_r 's and u_r 's work out as follows :

| | | | | | | |
|-------|-------|-------|---------|-----------|-------|---------|
| s_0 | s_1 | s_2 | \dots | s_{r-1} | s_r | \dots |
| + | 0 | 0 | \dots | 0 | + | \dots |
| | | | | | | |
| t_0 | t_1 | t_2 | \dots | t_{r-1} | t_r | \dots |
| 1 | 1 | 1 | \dots | 1 | 2 | \dots |
| | | | | | | |
| u_0 | u_1 | u_2 | \dots | u_{r-1} | u_r | \dots |
| 1 | 2 | 3 | \dots | r | $r+2$ | \dots |

which contradicts $u_r \leq r+1$.

Note that the two special cases $r = N-2$ and $r = N-1$ not covered by this argument cannot in any case arise because $s_0 + s_1 + \dots + s_{N-1} = 0$. □

Proposition 7.2

The substring $0+-+ \dots +-+0$ cannot appear within a string $s_0s_1\dots s_{N-1}$ generating a periodic sequence admissible adjacent to zero.

Proof

Consider a string $s_0s_1\dots s_{N-2}s_{N-1}$ in which $s_{N-1} = 0$. Then in addition to $t_{N-1} = 0$, we know also that $t_{N-2} = 0$, so that $x = u_{N-1} = u_{N-2} = u_{N-3}$. The cases $r = N-3$ and $r = N-1$ of the inequalities (7.14) specify that

$$-\frac{1}{2}(N-1) \leq y - \left(\frac{1}{2}(N-1) - (N-3)\right)x - Nx \leq \frac{1}{2}(N-1)$$

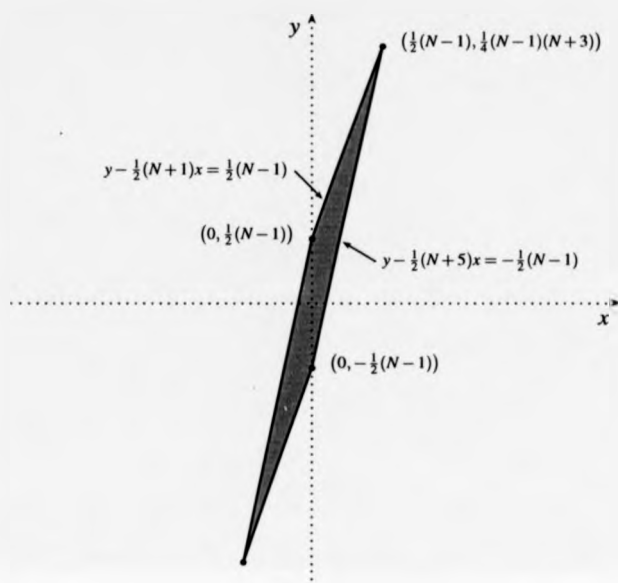


Figure 7.3: The corresponding region of the xy -plane for Proposition 7.2, determined by inequalities (7.21) and (7.22).

i.e.

$$-\frac{1}{2}(N-1) \leq y - \frac{1}{2}(N+5)x \leq \frac{1}{2}(N-1) \quad (7.21)$$

and

$$-\frac{1}{2}(N-1) \leq y - \frac{1}{2}(N+1)x \leq \frac{1}{2}(N-1). \quad (7.22)$$

The corresponding diagram is Figure 7.3.

Then for $r = 0, 1, \dots, N-4$,

$$\begin{aligned} Nu_r &\leq \frac{1}{4}(N-1)(N+3) - \frac{1}{2} \left(\frac{1}{2}(N-1) - r \right) (N-1) + \frac{1}{2}(N-1) \\ &= \frac{1}{2}(N-1)(r+3) \end{aligned} \quad (7.23)$$

and in this case the size of u_r is limited by

$$u_r \leq \frac{1}{2}(r+3)\frac{N-1}{N} < \frac{1}{2}(r+3). \quad (7.24)$$

If $0+-+ \dots +-+0$ appeared as a substring of $s_0s_1 \dots s_{N-1}$, by cycling we could suppose that $+-+ \dots +-+0$ appears at the start of the string and $s_{N-1} = 0$. But then the initial t_r 's and u_r 's work out as follows :

| | | | | | | | | | | | | |
|-------|-------|-------|-------|-------|-------|-------|-----|--------------------|--------------------|--------------------|--------------------|-----|
| s_0 | s_1 | s_2 | s_3 | s_4 | s_5 | s_6 | ... | s_{r-3} | s_{r-2} | s_{r-1} | s_r | ... |
| + | - | + | - | + | - | + | ... | + | - | + | 0 | ... |
| t_0 | t_1 | t_2 | t_3 | t_4 | t_5 | t_6 | ... | t_{r-3} | t_{r-2} | t_{r-1} | t_r | ... |
| 1 | 0 | 1 | 0 | 1 | 0 | 1 | ... | 1 | 0 | 1 | 1 | ... |
| u_0 | u_1 | u_2 | u_3 | u_4 | u_5 | u_6 | ... | u_{r-3} | u_{r-2} | u_{r-1} | u_r | ... |
| 1 | 1 | 2 | 2 | 3 | 3 | 4 | ... | $\frac{1}{2}(r-1)$ | $\frac{1}{2}(r-1)$ | $\frac{1}{2}(r+1)$ | $\frac{1}{2}(r+3)$ | ... |

which contradicts $u_r < \frac{1}{2}(r+3)$.

The case $r = N-3$ is also covered by this argument because $u_{N-3} = x$, and consequently $x \leq \frac{1}{2}(N-1) < \frac{1}{2}N = \frac{1}{2}((N-3)+3)$. The remaining case $r = N-2$ does not arise because $0+-+ \dots +-+0$ taken as the full string contradicts $s_0 + s_1 + \dots + s_{N-1} = 0$. \square

Remarks on the Propositions

1. The substring $++$ is the shortest length case of the substrings of the form $+0 \dots 0+$ and is among those substrings forbidden by Proposition 7.1. The substrings forbidden by Proposition 7.1 can be listed as $++$, $+0+$, $+00+$, $+000+$, etc., and those forbidden by Proposition 7.2 as $0+0$, $0+-+0$, $0+-+--0$, etc.
2. From Proposition 2.4 on the negation of strings it follows that substrings of the forms $-0 \dots 0-$ and $0-+-+ \dots -+-0$ are forbidden.

By means of these propositions we can establish a key feature of the composition of any string that generates a periodic sequence admissible adjacent to zero.

Corollary 7.3 (Format of generating strings)

The strings that generate periodic sequences admissible adjacent to zero can be cycled into the form

$$(+ -)^{a_1} 0^{b_1} (+ -)^{a_2} 0^{b_2} \dots (+ -)^{a_k} 0^{b_k}$$

or

$$(- +)^{a_1} 0^{b_1} (- +)^{a_2} 0^{b_2} \dots (- +)^{a_k} 0^{b_k},$$

where $2(a_1 + \dots + a_k) + (b_1 + \dots + b_k) = N$, where $a_i, b_i \geq 0$. That is, each such string is a run of $+-$ (or $-+$) pairs followed by a run of zeros, followed by a run of $+-$ (or $-+$) pairs, and so on.

Proof

Since $++$ and $--$ are forbidden by Proposition 7.1, the substrings occurring between the runs of zeros must alternate their $+$ and $-$ symbols, i.e. they are $+-+-\dots$ or $-+-+\dots$. In turn, Proposition 7.2 says these substrings can only be runs of $+-$ pairs or runs of $-+$ pairs. Finally Proposition 7.1 prevents mixing $+-$ pairs and $-+$ pairs in the same string.

Notice that here we also apply Propositions 7.1 and 7.2 to substrings produced by permitting wrap-around. That this is valid can be seen from two points of view : on the one hand the cyclical character of the Chua-Lin inequalities implicitly incorporates wrap-around, and as well if a string $s_0 s_1 \dots s_{N-1}$ generates an admissible periodic sequence then so does the length $2N$ string formed by concatenation, i.e. $s_0 s_1 \dots s_{N-1} s_0 s_1 \dots s_{N-1}$. \square

With the general format of the strings established, the task of classifying which strings generate periodic sequences admissible adjacent to zero, is to determine precisely the lengths of the various runs of 0's and $+-$ pairs within a string. Sections 7.4, 7.5 and 7.6 show how this information is extracted from the inequalities.

7.4 Ranges of x and y for strings generating periodic sequences admissible adjacent to zero

As noted in the previous section we can restrict attention to strings $s_0s_1 \dots s_{N-1}$ which may be written as combinations of runs of 0's and runs of +- pairs. By way of example consider the following string of length $N = 27$ and its associated scheme of t_r 's and u_r 's.

| r | 0 | 1 | 2 | 3 | ... | | | | | | | | | | | | | | | | | | | ... | 26 | | |
|-------|---|---|---|---|-----|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|-----|----|----|----|
| s_r | 0 | + | - | + | - | 0 | + | - | 0 | + | - | + | - | 0 | + | - | 0 | + | - | 0 | + | - | + | - | 0 | + | - |
| t_r | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | 1 | 0 |
| u_r | 0 | 1 | 1 | 2 | 2 | 2 | 3 | 3 | 3 | 4 | 4 | 5 | 5 | 5 | 6 | 6 | 6 | 7 | 7 | 7 | 8 | 8 | 9 | 9 | 9 | 10 | 10 |

We see here that t_r is 1 when $s_r = +$, but otherwise t_r is 0, and that correspondingly u_r counts the number of +'s from the start of the string to the r^{th} place. In particular $x = u_{N-1}$ is the count of the total number of +- pairs within the string : $x = 10$ in our example. Clearly these remarks are not restricted to our chosen example, but apply generally.

The range then for x consists of the integers from 1 to $\frac{1}{2}(N-1)$. We have excluded the possibility $x = 0$, corresponding to the trivial string consisting of 0's alone. The upper limit is set by the maximum number of +- pairs a string of length N can contain. The quantity $y = u_0 + u_1 + \dots + u_{N-1}$ is also an integer and is restricted by the inequalities

$$-\frac{1}{2}(N-1) \leq y - \frac{1}{2}(N+3)x \leq \frac{1}{2}(N-1)$$

and

$$-\frac{1}{2}(N-1) \leq y - \frac{1}{2}(N+1)x \leq \frac{1}{2}(N-1)$$

(i.e. the cases $r = N-2$ and $r = N-1$ of the inequalities (7.14)).

When the integer x between 1 and $\frac{1}{2}(N-1)$ is given, define integers y_{\max} and y_{\min} by

$$y_{\max} = \frac{1}{2}(N+1)x + \frac{1}{2}(N-1) \quad \text{and} \quad y_{\min} = \frac{1}{2}(N+3)x - \frac{1}{2}(N-1). \quad (7.25)$$

Then y is an integer in the range y_{\min} to y_{\max} : the possible (x, y) pairs are shown in Figure 7.4. The number of integer values between y_{\min} and y_{\max} is $y_{\max} - (y_{\min} - 1) = N - x$. For the string of length $N = 27$ given at the start of the section for which we saw $x = 10$, the formulae give $y_{\max} = 153$ and $y_{\min} = 137$, and adding the u_r 's we find $y = 142$.

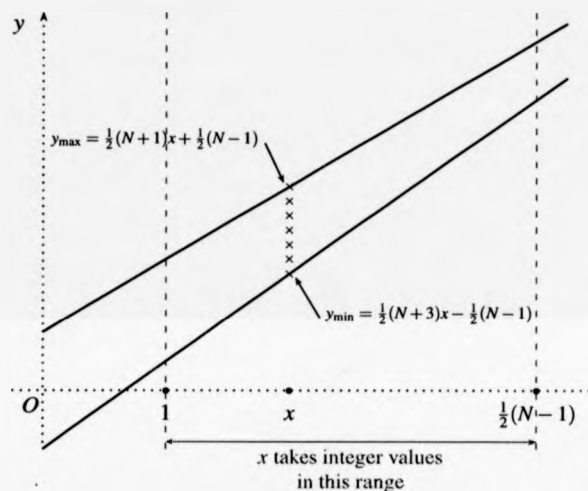


Figure 7.4: Location of the possible integer pairs (x, y) for a string generating a periodic sequence admissible adjacent to zero, cycled to commence with a $+-$ pair or a 0 .

For use later we describe how the y values in the range y_{\min} to y_{\max} permute when the original string $s_0 s_1 \dots s_{N-1}$ is cycled. If we cycle so as not to split a $+-$ pair, then x does not change. The effect on y depends on the initial digit of the string and there are two possibilities :

- (i) $s_0 = 0$ and $0s_1 s_2 \dots s_{N-1}$ cycles to $s_1 s_2 \dots s_{N-1} 0$ in which case y changes to $y + x$;
- (ii) $s_0 = +, s_1 = -$ and $+-s_2 \dots s_{N-1}$ cycles to $s_2 \dots s_{N-1} +-$ in which case y changes to $y - (N - 2x)$.

To prove (ii) we note that the value of y is reduced by :

- 2 for the $+-$ pair removed from the beginning of the string,
- $N - 2$ for the corresponding reduction by 1 of each of u_2 to u_{N-1} ,

and is augmented by :

- $2x$ for the $+-$ pair added to the end of the string.

7.5 Construction of a string for each choice of x between 1 and $\frac{1}{2}(N - 1)$

In this section we will show that for each x in the range $1 \leq x \leq \frac{1}{2}(N - 1)$, there corresponds a string $s_0 s_1 \dots s_{N-1}$, generating a periodic sequence admissible adjacent to zero. The string consists of runs of $+-$ pairs and runs of zeros, and the number of $+-$ pairs is precisely x .

Fix the integer x between 1 and $\frac{1}{2}(N - 1)$ and consider the inequalities (7.14), where y has been replaced by the specific value $y_{\max} = \frac{1}{2}(N + 1)x + \frac{1}{2}(N - 1)$. We find that this starting point will enable us to construct a string of the required type. With $y = y_{\max}$ the inequalities become

$$-\frac{1}{2}(N - 1) \leq \frac{1}{2}(N + 1)x + \frac{1}{2}(N - 1) - \left(\frac{1}{2}(N - 1) - r\right)x - Nu_r \leq \frac{1}{2}(N - 1)$$

which simplify to

$$(r + 1)x \leq Nu_r \leq (r + 1)x + (N - 1) \quad \text{for } r = 0, 1, \dots, N - 1. \quad (7.26)$$

Now $(r + 1)x, (r + 1)x + 1, (r + 1)x + 2, \dots, (r + 1)x + (N - 1)$ is a set of N consecutive integers, so precisely one of them is a multiple of N . Thus the inequalities (7.26) uniquely determine a progression of integers u_0, u_1, \dots, u_{N-1} and each u_r can be expressed as

$$u_r = \left\lceil \frac{(r + 1)x}{N} \right\rceil \quad \text{for } r = 0, 1, \dots, N - 1. \quad (7.27)$$

(Note that $\lceil \xi \rceil$ signifies the smallest integer greater than or equal to the real number ξ ; in the same way $\lfloor \xi \rfloor$ is the largest integer less than or equal to ξ .)

At this point we will change our viewpoint and take the equation (7.27) as the definition of a progression of u_r 's. Our aim now is to show that this progression of u_r 's corresponds (via the transformation scheme described in Section 7.2) to a string $s_0s_1 \dots s_{N-1}$ generating a periodic sequence admissible adjacent to zero, so consisting of runs of $+-$ pairs and runs of zeros, and which has precisely x runs of $+-$ pairs. We show this by establishing the following two statements :

- (a) The string $s_0s_1 \dots s_{N-1}$ coming from the u_r 's is legitimate, consists of runs $+-$ pairs and runs of zeros, and contains the correct number of $+-$ pairs, namely x such pairs.
- (b) The progression of u_r 's satisfies the inequalities (7.14).

In point (a) legitimacy is intended to signify that each digit in the string $s_0s_1 \dots s_{N-1}$ is one of $-$, 0 , or $+$: we do not know *a priori* that our progression of u_r 's necessarily originates in a string $s_0s_1 \dots s_{N-1}$ of the type we require. Point (b) is necessary because, although it is clear that the string $s_0s_1 \dots s_{N-1}$ and the u_r 's will be related in the appropriate fashion, we must satisfy the inequalities not with $y = y_{\max}$ but with $y = u_0 + u_1 + \dots + u_{N-1}$.

We start by collecting together the properties about the u_r 's and the corresponding t_r 's that will be needed to show that the string $s_0s_1 \dots s_{N-1}$ is of the required form.

Lemma 7.4

The progression of integers defined by $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$ has the following properties :

- (i) It is an increasing (in the weak sense) progression of N integers.
- (ii) $u_0 = u_1 = 1$.
- (iii) $u_{N-2} = u_{N-1} = x$.
- (iv) $u_{r+2} \leq u_r + 1$ (i.e. two consecutive u_r 's cannot both increment).

Proof

- (i) is immediate from the definition of the u_r 's.

(ii) $u_0 = \left\lceil \frac{x}{N} \right\rceil = 1$ because $\frac{x}{N}$ is less than 1.

$u_1 = \left\lceil \frac{2x}{N} \right\rceil$, and since $x \leq \frac{1}{2}(N-1)$, it follows that $\frac{2x}{N} \leq \frac{N-1}{N} < 1$ and hence $u_1 = 1$.

(iii) $u_{N-1} = \left\lceil \frac{Nx}{N} \right\rceil = \lceil x \rceil = x$.

$u_{N-2} = \left\lceil x - \frac{x}{N} \right\rceil$ and $x \leq \frac{1}{2}(N-1)$ implies $\frac{x}{N} < \frac{1}{2}$ so $u_{N-2} = x$.

(iv) Nu_r is the unique multiple of N in the range

$$(r+1)x, (r+1)x+1, \dots, (r+1)x+(N-1)$$

and Nu_{r+2} is the unique multiple of N in the range

$$(r+3)x, (r+3)x+1, \dots, (r+3)x+(N-1).$$

The number of integers from $(r+1)x$ to $(r+3)x+(N-1)$ is

$$\begin{aligned} ((r+3)x+N-1) - ((r+1)x-1) &= 2x+N \\ &\leq 2N-1 \quad \text{because } x \leq \frac{1}{2}(N-1). \end{aligned}$$

Nu_r and Nu_{r+2} both fall in this range so $Nu_{r+2} \leq Nu_r + N$, i.e. $u_{r+2} \leq u_r + 1$. \square

Lemma 7.5

The t_r 's obtained from the progression of u_r 's (via the transformation $t_0 = u_0$, $t_r = u_r - u_{r-1}$, as in (7.13)) have the following properties :

- (i) $t_0 = 1$ and $t_{N-1} = 0$.
- (ii) Each t_r is either 0 or 1.
- (iii) Two consecutive t_r 's cannot both be 1.

Proof

(i) $t_0 = u_0 = 1$ and $t_{N-1} = u_{N-1} - u_{N-2} = x - x = 0$.

(ii) Each t_r is at least zero because the u_r 's are increasing, and at most 1 because $t_{r+1} = u_{r+1} - u_r \leq u_{r+2} - u_r \leq 1$.

- (iii) $t_0 = 1$ and $t_1 = u_1 - u_0 = 0$, so t_0 and t_1 are not both 1. No other consecutive t_r 's can both be 1 because $t_{r+2} + t_{r+1} = u_{r+2} - u_r \leq 1$ for $r = 0, 1, \dots, N-3$. \square

The string $s_0 s_1 \dots s_{N-1}$ derived from the t_r 's via the scheme $s_0 = t_0$, $s_1 = t_1 - t_0$, ..., $s_{N-1} = t_{N-1} - t_{N-2}$ of (7.12) satisfies :

- (i) $s_0 = +$ and $s_1 = -$.
- (ii) The string $s_0 s_1 \dots s_{N-1}$ consists of runs of $+-$ pairs and runs of zeros.
- (iii) x is the count of the $+-$ pairs in the string $s_0 s_1 \dots s_{N-1}$.

These statements follow easily from the foregoing properties of the t_r 's and establish point (a) above.

There remains the task of showing that the string $s_0 s_1 \dots s_{N-1}$, produced in this way, generates a periodic sequence admissible adjacent to zero. It does so provided the Chua-Lin inequalities are satisfied. Now the u_r 's which arise in turn from the string $s_0 s_1 \dots s_{N-1}$ (via $t_r = s_0 + s_1 + \dots + s_r$ and $u_r = t_0 + t_1 + \dots + t_r$, as in Section 7.2) are clearly given by the original formulae

$$u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil \quad \text{for } r = 0, 1, \dots, N-1,$$

and we know that the Chua-Lin inequalities are equivalent to the inequalities (7.14) featuring the u_r 's, where $x = u_{N-1}$ and $y = u_0 + u_1 + \dots + u_{N-1}$. So we need to verify inequalities (7.14), essentially as we remarked in point (b) above, and this is accomplished by the next lemma.

Lemma 7.6

If $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$ for $r = 0, 1, \dots, N-1$, $x = u_{N-1}$ and $y = u_0 + u_1 + \dots + u_{N-1}$, then

$$-\frac{1}{2}(N-1) \leq y - \left(\frac{1}{2}(N-1) - r\right)x - Nu_r \leq \frac{1}{2}(N-1)$$

for $r = 0, 1, \dots, N-1$, i.e. inequalities (7.14) are satisfied.

Proof

We treat separately the two cases when x and N are coprime, and when x and N share a common factor.

x and N are coprime : In this case we show that if $y = u_0 + u_1 + \cdots + u_{N-1}$ then $y = y_{\max}$. The inequalities (7.14) are then established, because they coincide with the inequalities (7.26) which we know the u_r 's satisfy, for this was how the u_r 's were originally produced.

Each Nu_r is the unique multiple of N appearing in the set of N consecutive integers

$$(r+1)x, (r+1)x+1, \dots, (r+1)x+(N-1).$$

So we can write

$$Nu_r = (r+1)x + c_r \quad (7.28)$$

where each integer c_r is between 0 and $N-1$. Thus

$$\begin{aligned} N(u_0 + u_1 + \cdots + u_{N-1}) &= x(1+2+\cdots+N) + (c_0 + c_1 + \cdots + c_{N-1}) \\ &= \frac{1}{2}N(N+1)x + (c_0 + c_1 + \cdots + c_{N-1}). \end{aligned} \quad (7.29)$$

Now c_0, c_1, \dots, c_{N-1} are distinct (given that x and N are coprime), otherwise $c_r = c_{r'}$ and $r > r'$ implies

$$N(u_r - u_{r'}) = (r - r')x,$$

so that x divides $u_r - u_{r'} \leq x-1$, which is impossible.

Thus $c_0 + c_1 + \cdots + c_{N-1} = 0+1+\cdots+(N-1) = \frac{1}{2}N(N-1)$ and we obtain

$$u_0 + u_1 + \cdots + u_{N-1} = \frac{1}{2}(N+1)x + \frac{1}{2}(N-1) = y_{\max}. \quad (7.30)$$

x and N are not coprime : In this case let h be the highest common factor of x and N . Then we can write $x = ah$ and $N = mh$ where now a and m are coprime. The formula $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$ becomes $u_r = \left\lceil \frac{(r+1)a}{m} \right\rceil$ for $r = 0, 1, \dots, N-1$.

If $r' = m + r$ then

$$\begin{aligned} u_{r'} &= \left\lceil \frac{(m+r+1)a}{m} \right\rceil = \left\lceil a + \frac{(r+1)a}{m} \right\rceil \\ &= a + \left\lceil \frac{(r+1)a}{m} \right\rceil \\ &= a + u_r. \end{aligned} \quad (7.31)$$

This means that we can tabulate the values of the u_r 's as follows (where the entries in the individual rows correspond to the index ranges $0 \leq r \leq m-1$, $m \leq r \leq 2m-1$, ..., $(h-1)m \leq r \leq hm-1$):

$$\begin{array}{lll} u_0, & u_1, & \dots, u_{m-1}, \\ u_m = a + u_0, & u_{m+1} = a + u_1, & \dots, u_{2m-1} = a + u_{m-1}, \\ u_{2m} = 2a + u_0, & u_{2m+1} = 2a + u_1, & \dots, u_{3m-1} = 2a + u_{m-1}, \\ \vdots & \vdots & \ddots \\ u_{(h-1)m} = (h-1)a + u_0, & u_{(h-1)m+1} = (h-1)a + u_1, & \dots, u_{hm-1} = (h-1)a + u_{m-1}. \end{array}$$

Then

$$u_0 + u_1 + \dots + u_{N-1} = h(u_0 + u_1 + \dots + u_{m-1}) + (a + 2a + \dots + (h-1)a)m, \quad (7.32)$$

but we have shown in the case above that

$$u_0 + u_1 + \dots + u_{m-1} = \frac{1}{2}(m+1)a + \frac{1}{2}(m-1),$$

so

$$\begin{aligned} u_0 + u_1 + \dots + u_{N-1} &= h\left(\frac{1}{2}(m+1)a + \frac{1}{2}(m-1)\right) + \frac{1}{2}(h-1)ham \\ &= \frac{1}{2}h(amh + a + m - 1) \\ &= \frac{1}{2}(N+1)x + \frac{1}{2}(N-h). \end{aligned} \quad (7.33)$$

Note that in this case $u_0 + u_1 + \dots + u_{N-1}$ does not equal y_{\max} .

Finally we show that the u_r 's satisfy the inequalities (7.14) with $y = u_0 + u_1 + \dots + u_{N-1} = \frac{1}{2}(N+1)x + \frac{1}{2}(N-h)$. As we saw above, Nu_r is the unique multiple of N in the range

$$(r+1)x, (r+1)x+1, \dots, (r+1)x+(N-1).$$

There is also precisely one multiple of N in the range

$$(r+1)x-(h-1), (r+1)x-(h-1)+1, \dots, (r+1)x-(h-1)+(N-1)$$

and it must be the same multiple of N . Otherwise there would necessarily be a multiple of N in the range

$$(r+1)x-(h-1), \dots, (r+1)x-1.$$

But h cannot divide any integer in this latter range because it divides the integer $(r+1)x$ coming after, and the range contains only $(h-1)$ integers. Since h divides N there cannot be a multiple of N here.

This means that Nu_r satisfies

$$(r+1)x \leq Nu_r \leq (r+1)x+(N-h), \quad (7.34)$$

which certainly implies

$$(r+1)x - \frac{1}{2}(h-1) \leq Nu_r \leq (r+1)x + \frac{1}{2}(N-h) + \frac{1}{2}(N-1),$$

and, after rearrangement, we obtain

$$-\frac{1}{2}(N-1) \leq \left(\frac{1}{2}(N+1)x + \frac{1}{2}(N-h)\right) - \left(\frac{1}{2}(N-1) - r\right)x - Nu_r \leq \frac{1}{2}(N-1) \quad (7.35)$$

i.e.

$$-\frac{1}{2}(N-1) \leq y - \left(\frac{1}{2}(N-1) - r\right)x - Nu_r \leq \frac{1}{2}(N-1), \quad (7.36)$$

so the u_r 's satisfy the inequalities (7.14). □

7.6 Characterisation of the strings generating a periodic sequence admissible adjacent to zero

In the previous section we demonstrated how, given any integer x between 1 and $\frac{1}{2}(N-1)$, we can create a string $s_0s_1 \dots s_{N-1}$ generating a periodic sequence admissible adjacent to zero for which x is the number of $+-$ pairs. In this section we turn our attention to classifying all those strings generating a periodic sequence admissible adjacent to zero that contain x $+-$ pairs. Certainly the string produced by the construction described in the previous section is not the only such string : any cycled version of it which does not split a $+-$ pair will also generate a periodic sequence admissible adjacent to zero (albeit essentially the same sequence). One might reasonably expect there to be other strings generating a periodic sequence admissible adjacent to zero having x $+-$ pairs. This is not the case : the main result of this section shows that all the strings generating a periodic sequence admissible adjacent to zero and having x $+-$ pairs are cycled versions of the particular string produced in Section 7.5.

Our starting point is to use the y values in order to limit the number of strings generating a periodic sequence admissible adjacent to zero having a given value x as the count of $+-$ pairs. Recall that any string generating a periodic sequence admissible adjacent to zero and with x $+-$ pairs must have a y value between $y_{\min} = \frac{1}{2}(N+3)x - \frac{1}{2}(N-1)$ and $y_{\max} = \frac{1}{2}(N+1)x + \frac{1}{2}(N-1)$. The first result says that two strings with x $+-$ pairs having the same y value must be identical.

Proposition 7.7

Any two strings $s_0s_1 \dots s_{N-1}$ and $s'_0s'_1 \dots s'_{N-1}$, each generating a periodic sequence admissible adjacent to zero, that have the same values of x and y are identical strings.

Proof

Define the t_r 's and u_r 's for the string $s_0s_1 \dots s_{N-1}$ as previously. Then since $s_0s_1 \dots s_{N-1}$ generates a periodic sequence admissible adjacent to zero, the u_r 's satisfy the inequalities (7.14),

$$-\frac{1}{2}(N-1) \leq y - \left(\frac{1}{2}(N-1) - r\right)x - Nu_r \leq \frac{1}{2}(N-1)$$

for $r = 0, 1, \dots, N-1$, where $x = u_{N-1} = u_{N-2}$ and $y = u_0 + \dots + u_{N-1}$. We rearrange to obtain

$$y - \left(\frac{1}{2}(N-1) - r\right)x - \frac{1}{2}(N-1) \leq Nu_r \leq y - \left(\frac{1}{2}(N-1) - r\right)x + \frac{1}{2}(N-1),$$

so that the integer Nu_r lies between the two integers $y - \left(\frac{1}{2}(N-1) - r\right)x - \frac{1}{2}(N-1)$ and $y - \left(\frac{1}{2}(N-1) - r\right)x + \frac{1}{2}(N-1)$, and since this range consists of N integers, precisely one integer from the range is a multiple of N . Thus u_r can be written in terms of x and y by means of the formula

$$u_r = \left\lceil \frac{y - \left(\frac{1}{2}(N-1) - r\right)x - \frac{1}{2}(N-1)}{N} \right\rceil \quad \text{for } r = 0, 1, \dots, N-1. \quad (7.37)$$

When the same procedure is applied to the second sequence, $s'_0 s'_1 \dots s'_{N-1}$, we obtain an identical formula for u'_r in terms of $x = u'_{N-1}$ and $y = u'_0 + \dots + u'_{N-1}$.

Thus, if two strings have the same x and y values it follows that $u_r = u'_r$ for each r . But then $t_r = t'_r$ and $s_r = s'_r$ for each r , and the strings are identical. \square

We saw in Section 7.4 that for a given x value, there are $N - x$ values of y which lie between y_{\min} and y_{\max} . A consequence of Proposition 7.7 is that there are at most $N - x$ distinct strings generating a periodic sequence admissible adjacent to zero having x +- pairs.

The following proposition identifies a key structural difference between those strings generating a periodic sequence admissible adjacent to zero for which x and N are coprime and the others. Recognising this difference will be important for us in the proof of our main result, Proposition 7.9. Recall the terminology introduced in Section 2.4 that a string $s_0 s_1 \dots s_{N-1}$, which generates a periodic sequence, is *irreducible* if it is not a multiple copy of a shorter string; otherwise it is *reducible*.

Proposition 7.8

Let $s_0s_1 \dots s_{N-1}$ be a string generating a periodic sequence admissible adjacent to zero having x $+-$ pairs.

- (a) The string $s_0s_1 \dots s_{N-1}$ is irreducible precisely when x and N are coprime.
- (b) If x and N are not coprime and $h = \text{hcf}(x, N)$, $x = ah$, $N = mh$, then $s_0s_1 \dots s_{m-1}$ is irreducible and $s_0s_1 \dots s_{N-1}$ consists of h successive copies of $s_0s_1 \dots s_{m-1}$.

Proof

Statements (a) and (b) of the proposition follow from the two remarks :

- (i) Suppose $s_0s_1 \dots s_{N-1}$ is reducible and thus consists of h successive copies of the sub-string $s_0s_1 \dots s_{m-1}$, for some integers h and m . Then $N = mh$ and $x = u_{N-1} = ah$ where $a = u_{m-1}$. In particular if x and N are coprime $s_0s_1 \dots s_{N-1}$ is irreducible.
- (ii) Suppose x and N are not coprime, $h = \text{hcf}(x, N)$, $x = ah$ and $N = mh$. Then

$$\begin{aligned} \frac{y - \left(\frac{1}{2}(N-1) - (m+r)\right)x + \frac{1}{2}(N-1)}{N} &= \frac{y - \left(\frac{1}{2}(N-1) - r\right)x + \frac{1}{2}(N-1)}{N} + \frac{mx}{N} \\ &= \frac{y - \left(\frac{1}{2}(N-1) - r\right)x + \frac{1}{2}(N-1)}{N} + a, \end{aligned}$$

so from the formula (7.37) for u_r in terms of x and y , $u_{m+r} = a + u_r$. It follows from this that the string $s_0s_1 \dots s_{N-1}$ consists of h successive copies of $s_0s_1 \dots s_{m-1}$. Note also that $u_{m-1} = a$ because

$$u_{m-1} + (h-1)a = u_{(m-1)+m(h-1)} = u_{N-1} = x = ah,$$

so that from (i) we see that the string $s_0s_1 \dots s_{m-1}$ is irreducible. □

We are now in a position to prove our main result of this section, namely that the only strings with x $+-$ pairs which generate a periodic sequence admissible adjacent to zero are the string produced by the construction in Section 7.5, together with the cycled versions of it which do not split $+-$ pairs.

Proposition 7.9 (Characterisation of generating strings)

All strings $s_0s_1 \dots s_{N-1}$ generating a periodic sequence admissible adjacent to zero having $x \pm$ pairs are cycled versions of each other.

Proof

Case when x and N are coprime : Let $s_0s_1 \dots s_{N-1}$ be the string produced by the method of Section 7.5, for which $y = y_{\max}$. If we cycle $s_0s_1 \dots s_{N-1}$ so as not to split \pm pairs we obtain $N - x$ distinct strings, because $s_0s_1 \dots s_{N-1}$ is irreducible. We thus obtain $N - x$ distinct y values (by Proposition 7.7) and this accounts for all the y values between y_{\min} and y_{\max} .

Case when x and N share a factor : Set $h = \text{hcf}(x, N)$, $x = ah$ and $N = mh$. Proposition 7.8 shows that $s_0s_1 \dots s_{N-1}$ consists of h successive copies of $s_0s_1 \dots s_{m-1}$, and that this latter string is irreducible. Then $u_{m+r} = a + u_r$ and

$$\begin{aligned} y &= u_0 + u_1 + \dots + u_{N-1} = h(u_0 + \dots + u_{m-1}) + (a + 2a + \dots + (h-1)a)m \\ &= h(u_0 + \dots + u_{m-1}) + amh\left(\frac{1}{2}(h-1)\right) \end{aligned} \quad (7.38)$$

from which it can be seen that h divides y .

If we restrict attention just to the first part of the string, $s_0s_1 \dots s_{m-1}$, and the corresponding u_0, u_1, \dots, u_{m-1} , the irreducible case tells us (using the expressions (7.25) for y_{\min} and y_{\max}) that

$$\frac{1}{2}(m+3)a - \frac{1}{2}(m-1) \leq u_0 + \dots + u_{m-1} \leq \frac{1}{2}(m+1)a + \frac{1}{2}(m-1)$$

and moreover that $u_0 + \dots + u_{m-1}$ can take every integer value in this range (see the previous case, when x and N are coprime).

Corresponding to this range of values of $u_0 + \dots + u_{m-1}$, it follows from (7.38) that y comes between

$$h\left(\frac{1}{2}(m+3)a - \frac{1}{2}(m-1)\right) + \frac{1}{2}amh(h-1) = \frac{1}{2}(N+3)x - \frac{1}{2}(N-h) \quad (7.39)$$

and

$$h\left(\frac{1}{2}(m+1)a + \frac{1}{2}(m-1)\right) + \frac{1}{2}amh(h-1) = \frac{1}{2}(N+1)x + \frac{1}{2}(N-h), \quad (7.40)$$

and that y can assume every integer value which is a multiple of h in this range. There are $m-a = \frac{N-x}{h}$ values in all.

But the specific string produced by the construction in Section 7.5 (with the given N and x) has exactly $m-a$ distinct cycled versions, because the substring $s_0s_1 \dots s_{m-1}$ is irreducible and the full string is a multiple copy of it. \square

It is of interest to provide a formula for the number of periodic sequences of least period N whose interval of admissibility has left end-point zero. We have shown in this section that for a given string length N and for each integer x in the range 1 to $\frac{1}{2}(N-1)$ the only strings generating a periodic sequence admissible adjacent to zero and having exactly $x+-$ pairs are the string produced by the construction described in Section 7.5 and its cycled versions. When reckoning admissible periodic sequences recall that we have taken the view that all cycled versions of a generating string produce essentially the same periodic sequence. Consequently to count the number of periodic sequences admissible adjacent to zero having period N we can attribute one admissible sequence to each integer value of x in the range 1 to $\frac{1}{2}(N-1)$. However a string of length N which is reducible generates a sequence which duplicates one already arising from a substring, of shorter length and irreducible. Accordingly, when counting the number of admissible sequences of least period N , only those x coprime to N need be included in the count. To be consistent with our earlier computer searches it is also appropriate when counting the sequences to reintroduce negations : to each sequence arising within our classification there is a second (and distinct) sequence obtained from it by negating each digit.

Proposition 7.10

The number of periodic sequences of least period N , admissible in an interval having zero as left end-point, is $\phi(N)$ where ϕ is the Euler totient function.

Proof

Every x in the range $1, 2, \dots, \frac{1}{2}(N-1)$ which is coprime to N corresponds to an admissible sequence of least period N . The count of numbers coprime to N in the range $1, 2, \dots, \frac{1}{2}(N-1)$ is the same as the count of numbers coprime to N in the range $\frac{1}{2}(N-1) + 1, \frac{1}{2}(N-1) + 2, \dots, N-1$. This is because x is coprime to N if and only if $(N-x)$ is. But $\phi(N)$ is the count of numbers in the full range $1, 2, \dots, N-1$ coprime to N , so the count of numbers in $1, 2, \dots, \frac{1}{2}(N-1)$ coprime to N is $\frac{1}{2}\phi(N)$.

Finally, this quantity $\frac{1}{2}\phi(N)$ counts the number of admissible sequences originating from strings with $+-$ pairs. There are an equal number which originate from strings with $-+$ pairs (and none of these sequences duplicate any already counted), so the total number of admissible sequences is $\phi(N)$. \square

7.7 Examples and illustrations of the theory

A. Examples of the construction process

We present two examples to show how the construction of Section 7.5 gives rise in practice to the strings generating periodic sequences admissible adjacent to zero. As a first and simpler example, we consider the case $N = 7$ and $x = 2$ (so the string is to be of length 7 and have 2 $+-$ pairs). The formula (7.27) for the u_r 's becomes, with these specific values of N and x ,

$$u_r = \left\lceil \frac{2(r+1)}{7} \right\rceil \quad \text{for } r = 0, 1, \dots, 6,$$

from which the u_r 's can immediately be calculated; they are

$$\begin{aligned} u_0 &= \left\lceil \frac{2}{7} \right\rceil = 1 \\ u_1 &= \left\lceil \frac{4}{7} \right\rceil = 1 \\ u_2 &= \left\lceil \frac{6}{7} \right\rceil = 1 \\ u_3 &= \left\lceil \frac{8}{7} \right\rceil = 2 \\ u_4 &= \left\lceil \frac{10}{7} \right\rceil = 2 \\ u_5 &= \left\lceil \frac{12}{7} \right\rceil = 2 \\ u_6 &= \left\lceil \frac{14}{7} \right\rceil = 2. \end{aligned}$$

The string corresponding to this progression of u_r 's can be worked out from the expressions (7.12) and (7.13) relating the s_r 's, t_r 's and u_r 's. The following table shows that conversion :

| r | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|-------|---|---|---|---|---|---|---|
| u_r | 1 | 1 | 1 | 2 | 2 | 2 | 2 |
| t_r | 1 | 0 | 0 | 1 | 0 | 0 | 0 |
| s_r | + | - | 0 | + | - | 0 | 0 |

Thus the construction has produced the string $+-0+-00$, agreeing with the length 7 string having 2 $+-$ pairs found by our computer search.

By way of a larger scale example, we construct all the strings of length $N = 27$. Before beginning the calculation, note that although the formula $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$ is suited for producing the strings on computer, should that be desired, it is more convenient to work with an alternative version for calculations "by hand", as we do in this example. The inequalities

$$(r+1)x \leq Nu_r \leq (r+1)x + (N-1)$$

governing the u_r 's may be transformed to

$$\left\lceil \frac{N(u_r-1)+1}{x} \right\rceil \leq r+1 \leq \left\lfloor \frac{Nu_r}{x} \right\rfloor \quad (7.41)$$

since r is necessarily an integer.

It has been shown (Lemma 7.4) that the u_r 's form an increasing progression of integers, stepping up by at most 1 with each increment of r . Thus the values taken by the u_r 's are precisely the integers $1, 2, \dots, x$. The inequalities (7.41) give the ranges of r for which successively $u_r = 1, u_r = 2, \dots, u_r = x$. The following calculation finds the string corresponding to $N = 27$ and $x = 5$:

| | | | | |
|-----------|------|---|------|--------------------------------|
| $u_r = 1$ | when | $\left\lceil \frac{1}{5} \right\rceil \leq r+1 \leq \left\lfloor \frac{27}{5} \right\rfloor$ | i.e. | $r = 0, 1, 2, 3, 4$ |
| $u_r = 2$ | when | $\left\lceil \frac{28}{5} \right\rceil \leq r+1 \leq \left\lfloor \frac{54}{5} \right\rfloor$ | i.e. | $r = 5, 6, 7, 8, 9$ |
| $u_r = 3$ | when | $\left\lceil \frac{55}{5} \right\rceil \leq r+1 \leq \left\lfloor \frac{81}{5} \right\rfloor$ | i.e. | $r = 10, 11, 12, 13, 14, 15$ |
| $u_r = 4$ | when | $\left\lceil \frac{82}{5} \right\rceil \leq r+1 \leq \left\lfloor \frac{108}{5} \right\rfloor$ | i.e. | $r = 16, 17, 18, 19, 20$ |
| $u_r = 5$ | when | $\left\lceil \frac{109}{5} \right\rceil \leq r+1 \leq \left\lfloor \frac{135}{5} \right\rfloor$ | i.e. | $r = 21, 22, 23, 24, 25, 26$. |

[illegible]

Table 7.1: An extract from the computer search results : admissible periodic sequences generated by strings of length $N = 27$ at the parameter value $\theta = 0.1$

irreducible substring. For the final value $x = 9$, $\text{hcf}(x, N) = 9$, and the string is composed of 9 copies of a length 3 irreducible substring.

B. Locations of the strings in the xy -plane

Here we illustrate how the strings which generate periodic sequences admissible adjacent to zero, together with their cycled versions, are distributed throughout the parallelogram of allowable x and y values.

We saw in Section 7.3 that the allowable x and y values are the points on the integer lattice within the region determined by $-\frac{1}{2}(N-1) \leq y - \frac{1}{2}(N+3)x \leq \frac{1}{2}(N-1)$ and $-\frac{1}{2}(N-1) \leq y - \frac{1}{2}(N+1)x \leq \frac{1}{2}(N-1)$. Figures 7.5 and 7.6 show this distribution for the cases $N = 7$ and $N = 15$ respectively. Because it can be interpreted as the number of $+-$ pairs within a string, x is restricted to the range 1 to $\frac{1}{2}(N-1)$, so only these portions of the parallelograms have been drawn. Each cross on the diagram represents an integer pair (x, y) .

within the region, and alongside is printed the single string corresponding to that choice of x and y values.

Since $N = 7$ is prime, every x is coprime to N , and accordingly the theory of Section 7.4 guarantees a string for each of the $(N - x)$ integer y values between y_{\min} and y_{\max} . This is apparent in Figure 7.5.

The progression of y -values when one of the strings is cycled can be followed on the diagrams. Recall from Section 7.4 that y changes to $y + x$ if we cycle a **0** and to $y + (N - 2x)$ if we cycle a **+-** pair. Thus, for example, when $N = 7$ and $x = 2$ our construction produces the string **+ - 0 + - 0 0**, for which $y = y_{\max} = 11$. The results of cycling are as follows :

| string | y | |
|----------------------|-----|--|
| + - 0 + - 0 0 | 11 | |
| ↓ | | (cycle +- : y reduced by $N - 2x$) |
| 0 + - 0 0 + - | 8 | |
| ↓ | | (cycle 0 : y increased by x) |
| + - 0 0 + - 0 | 10 | |
| ↓ | | (cycle +- : y reduced by $N - 2x$) |
| 0 0 + - 0 + - | 7 | |
| ↓ | | (cycle 0 : y increased by x) |
| 0 + - 0 + - 0 | 9 | |
| ↓ | | (cycle 0 : y increased by x) |
| + - 0 + - 0 0 | 11 | (original string is restored) |

This accounts for the order in which we see the strings on the diagram as we look down a column corresponding to a fixed x .

The diagram for $N = 15$ exhibits all of these features, but includes also situations where x and N share a factor. As a result of Proposition 7.9, we know a string will exist only for those y values that are multiples of $h = \text{hcf}(x, N)$, and that come within the range $\frac{1}{2}(N + 3)x - \frac{1}{2}(N - h)$ to $\frac{1}{2}(N + 1)x + \frac{1}{2}(N - h)$. So for $N = 15$, when $x = 5$, h is 5 and strings are only present for the y values which are a multiple of 5 between 40 and 45, i.e. simply $y = 40$ and $y = 45$. Reference to Figure 7.6 shows that the two distinct cycled states of the string with $x = 5$, namely **0 + - 0 + - 0 + - 0 + -** and **+ - 0 + - 0 + - 0 + - 0**, do indeed have the y values 40 and 45.

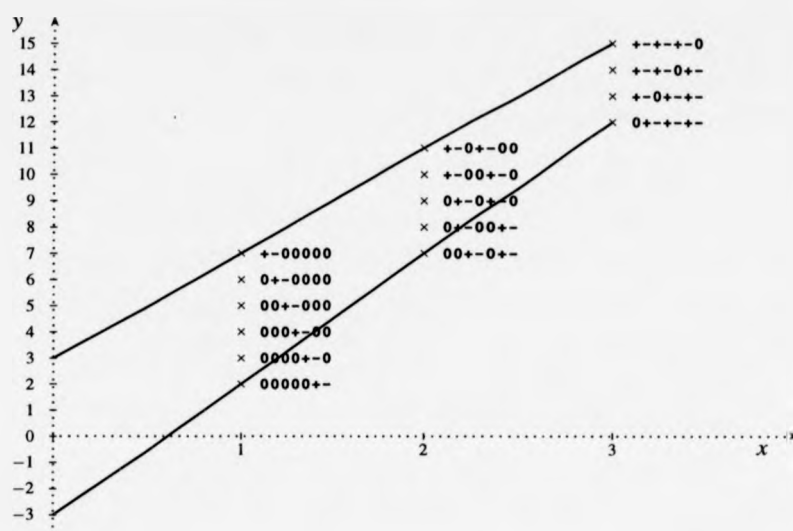


Figure 7.5: Location of the (x, y) integer pairs corresponding to the strings of length $N = 7$ that generate periodic sequences admissible adjacent to zero.

C. Table of short strings which generate periodic sequences admissible in an interval having zero as left end-point

We may at this stage use our classification to list the short period sequences (of odd period) which are admissible in an interval with left end-point zero. Table 7.2 shows the sequences with period less than 20. (Clearly, though, our method of classifying the strings allows us to tabulate such admissible sequences up to any period length desired.) The $\phi(N)$ admissible sequences having least period N have been listed by supplying in each instance a length N generating string. We have chosen to tabulate the strings produced by the construction of Section 7.5, together with their negations.

Table 7.2 may be compared with the list of admissible periodic sequences for $\theta = 0.15$ in Table 1 of Volume II, which was the result of computer searches. It will be noticed that the collection of sequences that have an interval of admissibility with left end-point zero, found empirically, coincides with the set of sequences guaranteed by our classification. Of

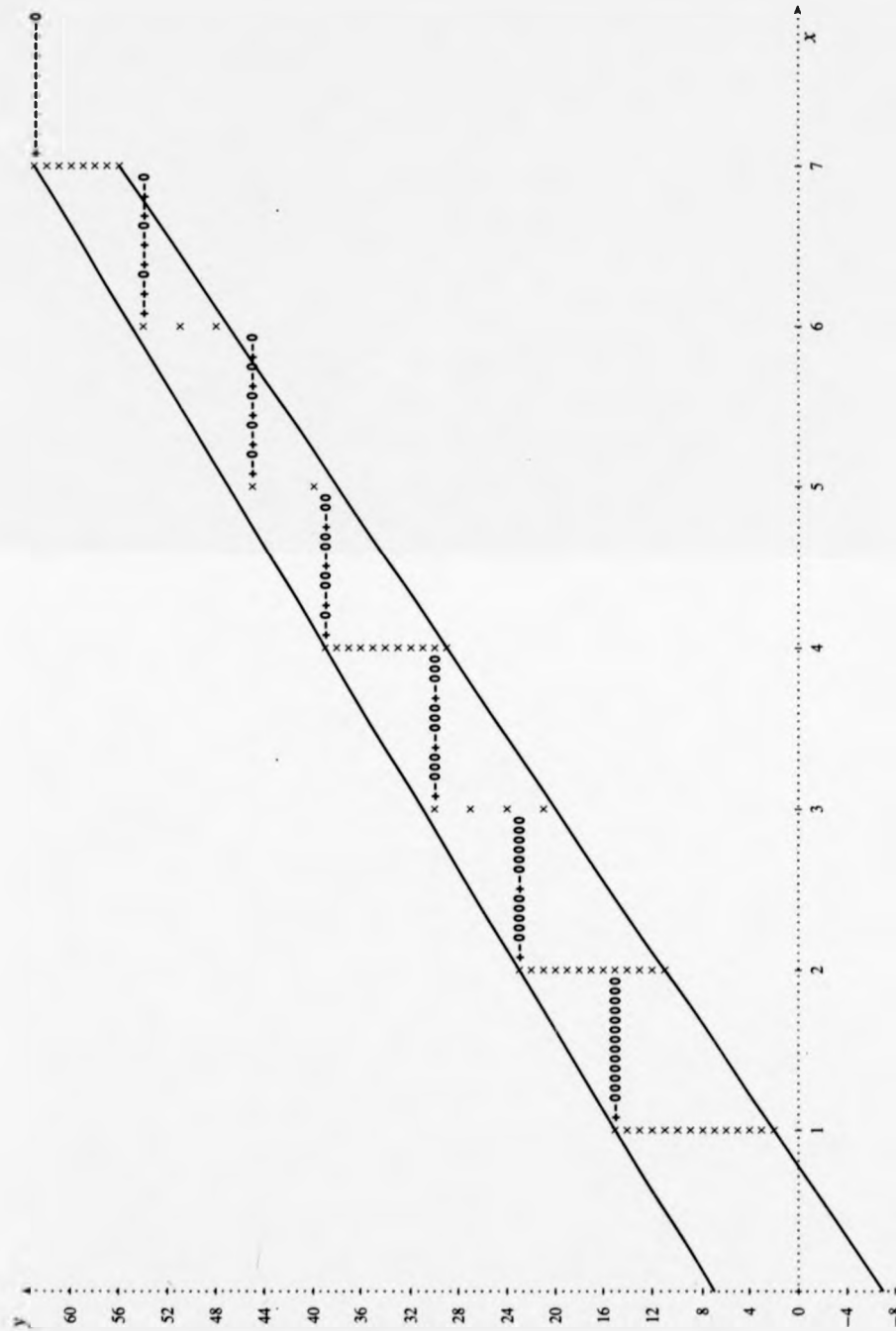


Figure 7.6: Location of the (x, y) pairs when $N = 15$. Only the string associated with the largest y value is labelled; the positions of their cycled versions are easily calculated from $y \mapsto y + x$ (cycling a 0) and $y \mapsto y - N + 2x$ (cycling a +- pair).

course, as a consequence of the strategy employed in the computer search, these sequences are listed in some random cycled form. With the insight gained through our theoretical investigation, we now have a canonical way of setting down the sequences, namely in terms of the cycled version which relates to the formula $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$.

| N | Number of Distinct Sequences, $\phi(N)$ | Generating Strings for the Admissible Sequences of Least Period N | |
|-----|---|--|--|
| 1 | 1 | 0 | |
| 3 | 2 | + - 0 | - + 0 |
| 5 | 4 | + - 000 + - + - 0 | - + 000 - + - + 0 |
| 7 | 6 | + - 00000 + - 0 + - 00 + - + - + - 0 | - + 00000 - + 0 - + 00 - + - + - + 0 |
| 9 | 6 | + - 0000000 + - 00 + - 000 + - + - + - + - 0 | - + 0000000 - + 00 - + 000 - + - + - + - + 0 |
| 11 | 10 | + - 000000000 + - 000 + - 0000 + - 0 + - 00 + - 00 + - + - 0 + - 0 + - 0 + - + - + - + - 0 | - + 000000000 - + 000 - + 0000 - + 0 - + 00 - + 00 - + - + 0 - + 0 - + 0 - + - + - + - + 0 |
| 13 | 12 | + - 00000000000 + - 0000 + - 00000 + - 00 + - 00 + - 000 + - 0 + - 0 + - 0 + - 00 + - + - 0 + - + - 0 + - 0 + - + - + - + - + - 0 | - + 00000000000 - + 0000 - + 00000 - + 00 - + 00 - + 000 - + 0 - + 0 - + 0 - + 00 - + - + 0 - + - + 0 - + 0 - + - + - + - + - + 0 |
| 15 | 8 | + - 0000000000000 + - 00000 + - 000000 + - 0 + - 00 + - 00 + - 00 + - + - + - + - + - + - 0 | - + 0000000000000 - + 00000 - + 000000 - + 0 - + 00 - + 00 - + 00 - + - + - + - + - + - + 0 |
| 17 | 16 | + - 000000000000000 + - 000000 + - 0000000 + - 000 + - 0000 + - 0000 + - 00 + - 00 + - 00 + - 000 + - 0 + - 0 + - 00 + - 0 + - 00 + - + - 0 + - 0 + - 0 + - 0 + - + - + - 0 + - + - 0 + - 0 + - + - + - + - + - + - 0 | - + 000000000000000 - + 000000 - + 0000000 - + 000 - + 0000 - + 0000 - + 00 - + 00 - + 00 - + 000 - + 0 - + 0 - + 00 - + 0 - + 00 - + - + 0 - + 0 - + 0 - + 0 - + - + + 0 - + - + 0 - + + 0 - + - + - + - + - + - + 0 |
| 19 | 18 | + - 00000000000000000 + - 0000000 + - 00000000 + - 0000 + - 0000 + - 00000 + - 00 + - 000 + - 000 + - 000 + - 0 + - 00 + - 00 + - 00 + - 00 + - 0 + - 0 + - 0 + - 0 + - 00 + - + - 0 + - 0 + - 0 + - 0 + - + - + - 0 + - + - 0 + - 0 + - + - + - + - + - + - 0 | - + 00000000000000000 - + 0000000 - + 00000000 - + 0000 - + 0000 - + 00000 - + 00 - + 000 - + 000 - + 000 - + 0 - + 00 - + 00 - + 00 - + 00 - + 0 - + 0 - + 0 - + 0 - + 00 - + - + 0 - + 0 - + 0 - + 0 - + - + + 0 - + - + 0 - + + 0 - + - + - + - + - + - + 0 |

Table 7.2: A complete list of the odd period sequences admissible in an interval having zero as left end-point, as far as least period $N = 19$, represented by their generating strings.

Chapter Summary

- A classification is achieved, when the period length is odd, of all periodic sequences having an interval of admissibility that extends on the left to $\theta = 0$. All their generating strings are described by a single arithmetic formula, and they account for the accumulation of sequences at values of θ close to zero noticed in our computer generated lists.
- These sequences "admissible adjacent to zero" are governed by a differential form of the Chua-Lin inequalities, in which the parameter θ no longer appears. Successive transformations by differencing schemes express the inequalities advantageously as

$$-\frac{1}{2}(N-1) \leq y - \left(\frac{1}{2}(N-1) - r\right)x - Nu_r \leq \frac{1}{2}(N-1)$$

for $r = 0, 1, \dots, N-1$, where $x = u_{N-1} = u_{N-2}$ and $y = u_0 + u_1 + \dots + u_{N-1}$.

- The determination of all odd-period periodic sequences admissible adjacent to zero then proceeds in three steps :
 - Allowing for cycling, their generating strings consist solely of runs of $+-$ pairs and runs of 0 's. (A $+-$ pair located where the otherwise stationary u_r increments.)
 - The strings of length N are classified by the quantity x , the count of the $+-$ pairs present. When N and x are given, the string $s_0s_1\dots s_{N-1}$ arising via $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$ generates a periodic sequence admissible adjacent to zero.
 - All other strings of length N having x $+-$ pairs that are admissible adjacent to zero are cycled versions of this string.
- A string produced by this scheme is irreducible precisely when N and x are coprime. Consequently the count of periodic sequences admissible adjacent to zero and of period length N is given in terms of the Euler totient function as $\phi(N)$.

Chapter 8

Periodic Sequences Admissible in an Interval Having Zero as Left End-point : Partial Classification of Even Length Generating Strings

We continue the investigation of periodic sequences having an interval of admissibility with left end-point zero. The concern now is with periodic sequences whose generating strings are of even length. Exactly as in the odd length case, we are able to account for the substantial collection of sequences appearing in the computer search lists for parameter values close to zero. However, the theoretical situation is not quite so clear cut as in the odd case in that the elimination of certain exotic possibilities, previously not feasible on grounds of parity, remains problematic.

8.1 Even length strings : relevance of the odd length methods

Recall that our starting point in the classification for N odd was to show, in Section 7.1, that since we are seeking intervals of admissibility adjacent to zero, we can specialise the Chua-

Lin inequalities by comparing the second derivatives of the f_r with those of $\pm g$ at 0. We showed that a string $s_0 s_1 \dots s_{N-1}$ which generates a periodic sequence admissible adjacent to zero satisfies the inequalities

$$\begin{aligned}
 -N &\leq \frac{N^2}{4}s_0 + \frac{(N-2)^2}{4}s_1 + \dots + \frac{(N-2)^2}{4}s_{N-1} \leq N \\
 -N &\leq \frac{N^2}{4}s_1 + \frac{(N-2)^2}{4}s_2 + \dots + \frac{(N-2)^2}{4}s_0 \leq N \\
 -N &\leq \frac{N^2}{4}s_2 + \frac{(N-2)^2}{4}s_3 + \dots + \frac{(N-2)^2}{4}s_1 \leq N \\
 &\vdots \\
 -N &\leq \frac{N^2}{4}s_{N-1} + \frac{(N-2)^2}{4}s_0 + \dots + \frac{(N-2)^2}{4}s_{N-2} \leq N
 \end{aligned} \tag{8.1}$$

(where the second and subsequent inequalities are obtained, from the one going before, by cycling the string $s_0 s_1 \dots s_{N-1}$), and satisfies also the equation $s_0 + s_1 + \dots + s_{N-1} = 0$. Reference to Section 7.1 shows that in establishing these inequalities, no use was made of the fact that N was odd, so the conclusions there hold equally for the case of strings with N even.

Our next step in the development was to strengthen these inequalities by showing that any string which satisfies the inequalities (8.1) must necessarily satisfy a modified version where the lower and upper bounds are sharpened to $-(N-1)$ and $(N-1)$ respectively, i.e.

$$\begin{aligned}
 -(N-1) &\leq \frac{N^2}{4}s_0 + \frac{(N-2)^2}{4}s_1 + \dots + \frac{(N-2)^2}{4}s_{N-1} \leq (N-1) \\
 -(N-1) &\leq \frac{N^2}{4}s_1 + \frac{(N-2)^2}{4}s_2 + \dots + \frac{(N-2)^2}{4}s_0 \leq (N-1) \\
 -(N-1) &\leq \frac{N^2}{4}s_2 + \frac{(N-2)^2}{4}s_3 + \dots + \frac{(N-2)^2}{4}s_1 \leq (N-1) \\
 &\vdots \\
 -(N-1) &\leq \frac{N^2}{4}s_{N-1} + \frac{(N-2)^2}{4}s_0 + \dots + \frac{(N-2)^2}{4}s_{N-2} \leq (N-1),
 \end{aligned} \tag{8.2}$$

and with $s_0 + s_1 + \dots + s_{N-1} = 0$.

However, the argument there that permitted this strengthening was based on parity considerations, and relied upon N being odd. Indeed we will show in the next section that in the case when N is even, there *do exist* strings which satisfy (8.1) but not (8.2).

For the moment though, we will restrict attention to strings $s_0 s_1 \dots s_{N-1}$ with N even which satisfy (8.2), and the rest of this section will show how the theory developed for the case when N is odd can be transferred to the present case with only minor modifications. In particular, we will be able to identify all the even length strings satisfying (8.2) and show that all of these strings generate periodic sequences admissible adjacent to zero.

Inspection of the argument given at the end of Section 7.1, to show that any string $s_0 s_1 \dots s_{N-1}$ which satisfies (8.2) generates a periodic sequence admissible adjacent to zero, reveals that it does not depend upon N being odd, so holds as well for the case of any even length string $s_0 s_1 \dots s_{N-1}$ which satisfies (8.2). Likewise, all of Section 7.2, where the variables t_r and u_r are introduced and their definitions given for a general string $s_0 s_1 \dots s_{N-1}$ satisfying (8.2), carries over without modification to the case when N is even.

Section 7.3 presented two propositions restricting the form of strings $s_0 s_1 \dots s_{N-1}$ which satisfy (8.2). Both hold for even N , and the rest of Section 7.3, showing that strings which generate periodic sequences admissible adjacent to zero consist of runs of $+-$ pairs and runs of zeros, depends only on these two propositions, and so carries over without change.

Proceeding as in Section 7.4, we see that for a string consisting of runs of $+-$ pairs and runs of zeros, $x = u_{N-1}$ is again the count of the number of $+-$ pairs but, as N is now even, its range is extended : x will be an integer in the range 1 to $\frac{1}{2}N$. The inequalities governing the region of allowable x and y values are unchanged, and as before, for a given x we define y_{\min} and y_{\max} by $y_{\min} = \frac{1}{2}(N+3)x - \frac{1}{2}(N-1)$ and $y_{\max} = \frac{1}{2}(N+1)x + \frac{1}{2}(N-1)$. Note though, that now y_{\min} and y_{\max} are integers only when x is odd. Consequently the number of integer values between y_{\min} and y_{\max} depends on the parity of x ; it equals $(N-x)$ when x is odd, and $(N-x-1)$ when x is even.

Section 7.5 applies to the case of even N , except for minor modifications to Lemma 7.4. Parts (i), (ii) and (iii) hold if we substitute $x \leq \frac{1}{2}N$ for $x \leq \frac{1}{2}(N-1)$. The proof of part (iv) relied upon $x \leq \frac{1}{2}(N-1)$, but extending it to include the case $x = \frac{1}{2}N$ presents no difficulty because when $x = \frac{1}{2}N$ the progression of u_r 's is simply 1, 1, 2, 2, 3, 3, ..., x , x for which $u_{r+2} = u_r + 1$. It then follows, just as for the case of odd N , that when N is even we have a

construction which produces a string satisfying the inequalities (8.2) (and hence generating a periodic sequence admissible adjacent to zero) for each x in the range $1, 2, \dots, \frac{1}{2}N$.

In similar fashion, the arguments of Section 7.6, showing that all strings which have the same value of x and which satisfy the inequalities (8.2) are cycled versions of one another, carry over to the case of even N without any modification. (Note that the fact that y_{\min} and y_{\max} are not integers when x is even is no inconsistency, because then x and N share the common factor 2 and in the non-coprime case we know there are no strings whose y values are y_{\min} or y_{\max} .)

In summary, then, we find that for integers x and N with $1 \leq x \leq \frac{1}{2}N$, the formula $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$ provides us with a string which generates a periodic sequence admissible adjacent to zero. This string consists of runs of $+-$ pairs and runs of zeros, and x counts the total number of $+-$ pairs. All other strings with the same values of x and N which satisfy the inequalities (8.2) are cycled versions of this one. When x and N are coprime the string is irreducible; otherwise if h is the highest common factor of x and N , then the string consists of h copies of a shorter irreducible string.

To consider now the wider question of characterising precisely which even length strings generate periodic sequences admissible adjacent to zero, we have shown that all the strings identified within the results of our computer searches as generators of periodic sequences admissible adjacent to zero can be obtained as instances of our construction. Thus the empirical evidence suggests that the only even length strings which generate periodic sequences admissible adjacent to zero are those which satisfy the strengthened inequalities, and so derive from our construction. To prove that this is so generally, it would be enough to rule out the strings which satisfy the inequalities (8.1) but not (8.2). It is this issue which we address in the following sections. At the outset of the presentation, though, it should be understood that a proof has not been achieved to establish in full generality that a string which satisfies (8.1) but not (8.2) cannot generate a periodic sequence admissible adjacent to zero.

We address the task in several stages. In Section 8.2 we identify all even length strings which satisfy inequalities (8.1) but not (8.2). We take up the harder part of the task, that

of disallowing these strings, in Section 8.3, where we show that information about higher derivatives of the functions f_r is needed to decide that one of these strings cannot generate a periodic sequence admissible adjacent to zero. As a consequence we isolate, in Section 8.4, a specific combinatorial problem responsible for the obstruction to a proof of the general result. In Section 8.5 we do resolve the issue of the precise classification of the strings that generate periodic sequences admissible adjacent to zero when N is twice a prime, and we work some way towards its resolution for other even N .

8.2 Strings satisfying the original but not the strengthened inequalities

As we indicated in the previous section, when N is even there is an extra obstacle to classifying the strings which generate periodic sequences admissible adjacent to zero, namely that there exist strings which satisfy the inequalities (8.1), but not their strengthened form (8.2). An example will make this clear : consider the length $N = 10$ string $+ - 0 0 + - 0 0 0 0$. Denoting by f_r'' the second derivative of the function $f_r(\theta)$ evaluated at $\theta = 0$, we find for this string :

$$\begin{aligned}
 f_0'' &= -(5^2 s_0 + 4^2 s_1 + 3^2 s_2 + 2^2 s_3 + 1^2 s_4 + 1^2 s_6 + 2^2 s_7 + 3^2 s_8 + 4^2 s_9) = -10, \\
 f_1'' &= -(5^2 s_1 + 4^2 s_2 + 3^2 s_3 + 2^2 s_4 + 1^2 s_5 + 1^2 s_7 + 2^2 s_8 + 3^2 s_9 + 4^2 s_0) = 6, \\
 f_2'' &= -(5^2 s_2 + 4^2 s_3 + 3^2 s_4 + 2^2 s_5 + 1^2 s_6 + 1^2 s_8 + 2^2 s_9 + 3^2 s_0 + 4^2 s_1) = 2, \\
 f_3'' &= -(5^2 s_3 + 4^2 s_4 + 3^2 s_5 + 2^2 s_6 + 1^2 s_7 + 1^2 s_9 + 2^2 s_0 + 3^2 s_1 + 4^2 s_2) = -2, \\
 f_4'' &= -(5^2 s_4 + 4^2 s_5 + 3^2 s_6 + 2^2 s_7 + 1^2 s_8 + 1^2 s_0 + 2^2 s_1 + 3^2 s_2 + 4^2 s_3) = -6, \\
 f_5'' &= -(5^2 s_5 + 4^2 s_6 + 3^2 s_7 + 2^2 s_8 + 1^2 s_9 + 1^2 s_1 + 2^2 s_2 + 3^2 s_3 + 4^2 s_4) = 10, \\
 f_6'' &= -(5^2 s_6 + 4^2 s_7 + 3^2 s_8 + 2^2 s_9 + 1^2 s_0 + 1^2 s_2 + 2^2 s_3 + 3^2 s_4 + 4^2 s_5) = 6, \\
 f_7'' &= -(5^2 s_7 + 4^2 s_8 + 3^2 s_9 + 2^2 s_0 + 1^2 s_1 + 1^2 s_3 + 2^2 s_4 + 3^2 s_5 + 4^2 s_6) = 2, \\
 f_8'' &= -(5^2 s_8 + 4^2 s_9 + 3^2 s_0 + 2^2 s_1 + 1^2 s_2 + 1^2 s_4 + 2^2 s_5 + 3^2 s_6 + 4^2 s_7) = -2, \\
 f_9'' &= -(5^2 s_9 + 4^2 s_0 + 3^2 s_1 + 2^2 s_2 + 1^2 s_3 + 1^2 s_5 + 2^2 s_6 + 3^2 s_7 + 4^2 s_8) = -6.
 \end{aligned}$$

The inequalities (8.1) are precisely $-N \leq f_r'' \leq N$, for $r = 0, 1, \dots, N-1$, so we see at once that the string under consideration here satisfies inequalities (8.1); but it does not satisfy the strengthened inequalities (8.2) because of f_0'' and f_5'' , both have absolute value 10, the value of N .

When considering a string $s_0 s_1 \dots s_{N-1}$ which satisfies the inequalities (8.1) but not (8.2), we can, without loss of generality, suppose $f_0'' = -N$. This is because any string which satisfies (8.1) but not (8.2) must have at least one $f_r'' = \pm N$. By cycling and negation we may assume that $f_0'' = -N$.

For the string $s_0 s_1 \dots s_{N-1}$, with $s_0 + s_1 + \dots + s_{N-1} = 0$, we define as previously, the variables t_r and u_r (see Section 7.2). Then

$$\begin{aligned} f_0'' &= -2(u_0 + u_1 + \dots + u_{N-1}) + (N+1)u_{N-1} \\ &= (N+1)x - 2y \end{aligned} \quad (8.3)$$

if $x = u_{N-2} = u_{N-1}$ and $y = u_0 + u_1 + \dots + u_{N-1}$, and similarly for $r = 0, 1, \dots, N-2$,

$$f_{r+1}'' = ((N-1) - 2r)x + 2Nu_r - 2y. \quad (8.4)$$

The inequalities (8.1) can then be expressed in terms of the u_r variables as

$$-N \leq ((N-1) - 2r)x + 2Nu_r - 2y \leq N \quad (8.5)$$

for $r = 0, 1, \dots, N-1$. As before, two of these inequalities involve x and y alone, and together determine a parallelogram containing the integer pairs (x, y) that satisfy the inequalities. However, since we are now considering strings satisfying (8.1) and $f_0'' = -N$, we are further restricted to values for x and y which correspond to a point on the side of the parallelogram defined by the equation $y = \frac{1}{2}(N+1)x + \frac{1}{2}N$. Figure 8.1 shows the parallelogram with this side highlighted. It can at once be seen from the diagram that x must lie between 0 and N .

First note that there are no integer pairs (x, y) with x odd for which $f_0'' = -N$, because

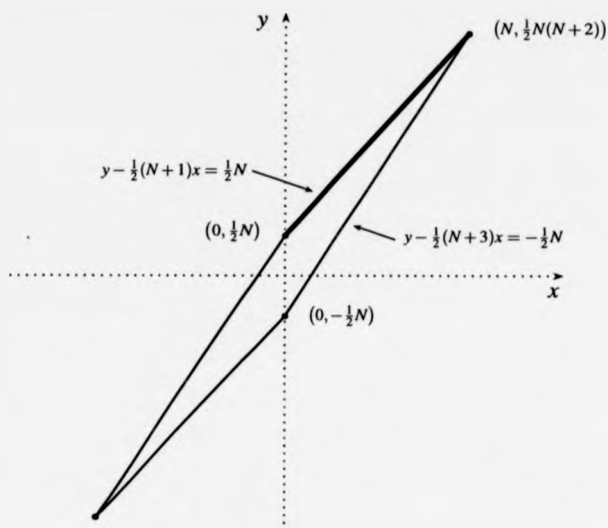


Figure 8.1: Region of the xy -plane determined by cases $r = N - 2$ and $r = N - 1$ of the inequalities (8.5). Integer pairs (x, y) on the edge drawn in bold correspond to the strings for which $f_0'' = -N$.

the condition $f_0'' = -N$ implies $2y = N + (N + 1)x$, so that y is an integer only if x is even. Equivalently, any string $s_0 s_1 \dots s_{N-1}$ that satisfies the inequalities (8.1) and for which $x = u_{N-1}$ is odd will automatically satisfy the inequalities (8.2). For if we take the inequalities (8.1) written as above in terms of the u_r 's,

$$-N \leq (N - 1)x - 2rx + 2Nu_r - 2y \leq N,$$

then, when x is odd, each summand of the middle term is even except $(N - 1)x$. So $(N - 1)x - 2rx + 2Nu_r - 2y$ is odd, and the extreme terms $\pm N$ can be replaced by $\pm(N - 1)$.

We do not look for solutions of the inequalities (8.1) with $f_0'' = -N$ when either $x = 0$ or $x = N$. In these two cases strings do exist which satisfy (8.1) but not (8.2). However, as we shall later see, (in Section 8.5), other tools are available that allow us to exclude $x = 0$ or $x = N$ when the string generates a periodic sequence admissible adjacent to zero.

We are then concerned here with strings $s_0 s_1 \dots s_{N-1}$, satisfying the inequalities (8.5), for which $f_0'' = -N$, and $x = 2, 4, \dots, N-2$. To identify these we use an approach similar to that of our construction in Section 7.5. The condition $f_0'' = -N$ implies $y = \frac{1}{2}(N+1)x + \frac{1}{2}N$, and substituting this value into each of the inequalities (8.5) we find

$$-N \leq (r+1)x - Nu_r \leq 0$$

for $r = 0, 1, \dots, N-2$. Expressing u_r in terms of x gives

$$\left\lceil \frac{(r+1)x}{N} \right\rceil \leq u_r \leq \left\lfloor \frac{(r+1)x}{N} \right\rfloor + 1 \quad \text{for } r = 0, 1, \dots, N-2;$$

and of course $u_{N-1} = x$.

Now, N and x are even, so we may write $N = 2mh$, $x = 2ah$ where $h = \text{hcf}(\frac{1}{2}x, \frac{1}{2}N)$, the integers m and a are coprime, and $a < m$. The inequalities for the u_r 's are then

$$\left\lceil \frac{(r+1)a}{m} \right\rceil \leq u_r \leq \left\lfloor \frac{(r+1)a}{m} \right\rfloor + 1 \quad \text{for } r = 0, 1, \dots, N-2; \quad (8.6)$$

and $u_{N-1} = x$. By contrast with the formula we were able to derive in Section 7.5, now not all of the u_r 's are determined uniquely by these conditions. The following example, where we take $N = 42$ and $x = 18$, makes the point clear.

Example 8.1 : Progressions of u_r 's satisfying (8.5) for $N = 42$ and $x = 18$

$$\left. \begin{array}{l} N = 42 = 2 \times 7 \times 3 \\ x = 18 = 2 \times 3 \times 3 \end{array} \right\} \text{ so } h = 3, m = 7 \text{ and } a = 3.$$

For this example the inequalities (8.6) are

$$\left\lceil \frac{3(r+1)}{7} \right\rceil \leq u_r \leq \left\lfloor \frac{3(r+1)}{7} \right\rfloor + 1 \quad \text{for } r = 0, 1, \dots, 40,$$

and the possible values for each of the u_r 's are shown in the following table.

| r | u_r | r | u_r | r | u_r |
|-----|--------|-----|----------|-----|----------|
| 0 | 1 | 14 | 7 | 28 | 13 |
| 1 | 1 | 15 | 7 | 29 | 13 |
| 2 | 2 | 16 | 8 | 30 | 14 |
| 3 | 2 | 17 | 8 | 31 | 14 |
| 4 | 3 | 18 | 9 | 32 | 15 |
| 5 | 3 | 19 | 9 | 33 | 15 |
| 6 | 3 or 4 | 20 | 9 or 10 | 34 | 15 or 16 |
| 7 | 4 | 21 | 10 | 35 | 16 |
| 8 | 4 | 22 | 10 | 36 | 16 |
| 9 | 5 | 23 | 11 | 37 | 17 |
| 10 | 5 | 24 | 11 | 38 | 17 |
| 11 | 6 | 25 | 12 | 39 | 18 |
| 12 | 6 | 26 | 12 | 40 | 18 |
| 13 | 6 or 7 | 27 | 12 or 13 | 41 | 18 |

Whenever $(r + 1)$ is a multiple of m there is a choice for u_r between two consecutive integers (except that $u_{N-1} = x$); otherwise the inequalities (8.6) determine the value of u_r uniquely. But $y = \frac{1}{2}(N + 1)x + \frac{1}{2}N$ represents also the sum of all the u_r 's. The following lemma describes how this fact restricts the assignment of values for the u_r 's where there is a choice between the higher and lower levels.

Lemma 8.1

Of the u_r 's for which a choice of values arises, i.e. $(r + 1)$ is a multiple of m , precisely h of them are set at the higher level.

Proof

For the moment, whenever there is a choice of value for one of the u_r 's, select the lower level. This means that $u_{m-1} = a$, $u_{2m-1} = 2a$, etc. Quite generally then the u_r 's are specified by the formula $u_r = \left\lceil \frac{(r+1)a}{m} \right\rceil$ for $r = 0, 1, \dots, N - 1$. But we showed, in the course of the proof of Lemma 7.6 that when this is so we have

$$u_0 + u_1 + \dots + u_{N-1} = \frac{1}{2}(N + 1)x + \frac{1}{2}(N - 2h)$$

(where of course $2h$ is now the highest common factor of x and N). Because this quantity falls short of $y = \frac{1}{2}(N + 1)x + \frac{1}{2}N$ by h , precisely h of the u_r 's, where there is a choice, have to be at the higher level. \square

Still considering strings with $f_0'' = -N$, for given values of N and x , the number of distinct progressions for the u_r 's satisfying the inequalities (8.5) is $\binom{2h-1}{h}$. For Example 8.1 where $N = 42$ and $x = 18$, there are thus $\binom{5}{3} = 10$ possibilities, and they correspond to the following schemes of choices of lower (L) or higher (H) values for each of $u_6, u_{13}, u_{20}, u_{27}$ and u_{34} :

| | u_6 | u_{13} | u_{20} | u_{27} | u_{34} |
|-----|-------|----------|----------|----------|----------|
| 1. | L | L | H | H | H |
| 2. | L | H | L | H | H |
| 3. | L | H | H | L | H |
| 4. | L | H | H | H | L |
| 5. | H | L | L | H | H |
| 6. | H | L | H | L | H |
| 7. | H | L | H | H | L |
| 8. | H | H | L | L | H |
| 9. | H | H | L | H | L |
| 10. | H | H | H | L | L |

The strings corresponding to these ten schemes are as follows :

| | |
|-----|---|
| 1. | + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 |
| 2. | + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 |
| 3. | + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 |
| 4. | + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 |
| 5. | + - + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 |
| 6. | + - + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 |
| 7. | + - + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 |
| 8. | + - + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 |
| 9. | + - + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 |
| 10. | + - + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 + - + - + - 0 |

and each of these satisfies the inequalities (8.1) but not (8.2).

Now that we have identified all the strings which satisfy (8.1) with $f_0'' = -N$, we consider for the benefit of our subsequent investigation the character of certain cycled versions of these strings. Starting then with a string which satisfies (8.1) and for which $f_0'' = -N$, if

we substitute $y = \frac{1}{2}(N+1)x + \frac{1}{2}N$ into equation (8.4) we find

$$f''_{r+1} = -2(r+1)x + 2Nu_r - N. \quad (8.7)$$

At those positions where (8.6) permits a choice and u_r takes the lower value, so $u_r = \frac{(r+1)x}{N}$, and $f''_{r+1} = -N$. For the string $s_0s_1 \dots s_{N-1}$, satisfying (8.1) and with $f''_0 = -N$, we find exactly h of the f''_r 's assume the value $-N$. Suppose we cycle the string to obtain the string $\tilde{s}_0\tilde{s}_1 \dots \tilde{s}_{N-1}$, where say $\tilde{s}_0 = s_{r+1}$, $\tilde{s}_1 = s_{r+2}$, etc., then $\tilde{f}''_0 = f''_{r+1}$ (recall the formula for f''_{r+1} in terms of s_0, s_1, \dots, s_{N-1}). This means that if we cycle the original string to bring into the first place s_{r+1} associated with a position where u_r can and does take a lower value, then the cycled version has its own $\tilde{f}''_0 = -N$.

Continuing with such a cycled version of the original string (which satisfies (8.1) and has $f''_0 = -N$), the associated $\tilde{u}_0, \tilde{u}_1, \dots, \tilde{u}_{N-1}$ themselves conform to the inequalities (8.6) so that

$$\left\lceil \frac{(r+1)a}{m} \right\rceil \leq \tilde{u}_r \leq \left\lfloor \frac{(r+1)a}{m} \right\rfloor + 1 \quad \text{for } r = 0, 1, \dots, N-2,$$

and $\tilde{u}_{N-1} = x$. Thus, apart from the $(2h-1)$ positions where there is a choice between upper and lower levels, \tilde{u}_r coincides with u_r of the original string. To decide at the $(2h-1)$ ambiguous positions, recall that u_r assumes its lower or upper value according to whether $f''_{r+1} = -N$ or $f''_{r+1} = N$. But since the f''_r 's cycle in step with the s_r 's, so does the scheme of choices allotting a higher or lower value to the ambiguous u_r 's.

The manner in which this scheme of choices cycles with the string is best illustrated by an example. Consider the second string in the example above (with $N = 42$ and $x = 18$), where the scheme of choices is L H L H H, relating to the determination of $u_6, u_{13}, u_{20}, u_{27}$ and u_{34} respectively. We have seen that this scheme of choices equivalently determines which of $f''_7, f''_{14}, f''_{21}, f''_{28}$ and f''_{35} have the value $-N$ and which have the value N . When describing how the scheme of choices cycles, it helps to include f''_0 as an extra member of the scheme. Since $f''_0 = -N$, this always adds an extra L, and we will write it in parentheses as (L) at the front to distinguish it from other terms which can be either L or H. With this

addition, the scheme of choices for our example is

$$\begin{array}{cccccc} f_0'' & f_7'' & f_{14}'' & f_{21}'' & f_{28}'' & f_{35}'' \\ (L) & L & H & L & H & H. \end{array}$$

The schemes of choices (and hence the progressions of the u_r 's) for cycled versions of the string, each having its $f_0'' = -N$, are then obtained by simply cycling this extended scheme of choices but requiring always an L in the first place (corresponding to f_0''). So there are three such cycled versions for the string we selected, and their schemes of choices are

$$\begin{array}{cccccc} f_0'' & f_7'' & f_{14}'' & f_{21}'' & f_{28}'' & f_{35}'' \\ (L) & L & H & L & H & H \\ (L) & H & L & H & H & L \\ (L) & H & H & L & L & H \end{array}$$

8.3 Exclusion by inspection of higher derivatives

With the strings satisfying the inequalities (8.1) but not (8.2) identified, the task now is to show that these same strings do not generate periodic sequences admissible adjacent to zero, and for this we need to refer back to the Chua-Lin inequalities. To decide admissibility adjacent to zero for these strings, because f_0'' has the same value as $(-g)''(0)$, we examine higher derivatives. The third derivatives of the f_r 's and of g all vanish at zero, exactly as we found the first derivatives did. So to obtain a new test for admissibility we have to consider fourth derivatives.

Using the Taylor series expansions of $f_0(\theta)$ and $g(\theta)$, in a similar fashion to that presented in Section 7.1, and noting the values at $\theta = 0$ of $f_0(\theta)$ and $g(\theta)$ and of their first three derivatives, i.e.

$$\begin{array}{ll} f_0 & = 0 & g & = 0 \\ f_0' & = 0 & g' & = 0 \\ f_0'' & = -N & g'' & = N \\ f_0^{(3)} & = 0 & g^{(3)} & = 0 \end{array}$$

it follows that a sufficient condition for $f_0(\theta) < (-g)(\theta) < 0$ throughout some interval with left end-point zero (and hence a sufficient condition for the string *not* to generate a periodic sequence) is

$$f_0^{(4)} < (-g)^{(4)}. \quad (8.8)$$

The value of $(-g)^{(4)}$ in terms of the string length N is

$$(-g)^{(4)} = 2N \left(\frac{N^2}{4} + 1 \right). \quad (8.9)$$

We can now resume the example presented at the beginning of Section 8.2, involving the string $+ - 00 + - 0000$ of length $N = 10$. For this string,

$$f_0^{(4)} = 5^4 s_0 + 4^4 s_1 + 3^4 s_2 + 2^4 s_3 + 1^4 s_4 + 1^4 s_6 + 2^4 s_7 + 3^4 s_8 + 4^4 s_9 = 370,$$

but $(-g)^{(4)} = 520$. Thus $f_0^{(4)} < (-g)^{(4)}$, and so this string does not generate a periodic sequence admissible adjacent to zero.

To proceed we need a general formula for $f_0^{(4)}$, applicable to any string $s_0 s_1 \dots s_{N-1}$. Differentiating the expression for $f_0(\theta)$ and setting $\theta = 0$, we obtain

$$\begin{aligned} f_0^{(4)} &= \left(\frac{N}{2}\right)^4 s_0 + \left(\frac{N-2}{2}\right)^4 s_1 + \dots + \left(\frac{N-2}{2}\right)^4 s_{N-1} \\ &= \frac{1}{16} \sum_{r=0}^{N-1} (N-2r)^4 s_r. \end{aligned} \quad (8.10)$$

Rewritten in terms of the t_r 's and the u_r 's, corresponding formulae are

$$\begin{aligned} f_0^{(4)} &= \left(\frac{N}{2}\right)^4 t_0 + \left(\frac{N-2}{2}\right)^4 (t_1 - t_0) + \dots + \left(\frac{N-2}{2}\right)^4 (-t_{N-2}) \\ &= \left(\left(\frac{N}{2}\right)^4 - \left(\frac{N-2}{2}\right)^4\right) t_0 + \left(\left(\frac{N-2}{2}\right)^4 - \left(\frac{N-4}{2}\right)^4\right) t_1 + \dots + \left(\left(\frac{N-4}{2}\right)^4 - \left(\frac{N-2}{2}\right)^4\right) t_{N-2} \\ &= \frac{1}{16} \sum_{r=0}^{N-2} \{(N-2r)^4 - (N-2(r+1))^4\} t_r, \end{aligned} \quad (8.11)$$

and

$$\begin{aligned}
 f_0^{(4)} &= \left(\left(\frac{N}{2} \right)^4 - \left(\frac{N-2}{2} \right)^4 \right) u_0 + \left(\left(\frac{N-2}{2} \right)^4 - \left(\frac{N-4}{2} \right)^4 \right) (u_1 - u_0) + \dots + \\
 &\quad \left(\left(\frac{N-4}{2} \right)^4 - \left(\frac{N-2}{2} \right)^4 \right) (u_{N-2} - u_{N-3}) \\
 &= \left(\left(\frac{N}{2} \right)^4 - 2 \left(\frac{N-2}{2} \right)^4 + \left(\frac{N-4}{2} \right)^4 \right) u_0 + \left(\left(\frac{N-2}{2} \right)^4 - 2 \left(\frac{N-4}{2} \right)^4 + \left(\frac{N-6}{2} \right)^4 \right) u_1 + \dots + \\
 &\quad \left(\left(\frac{N-6}{2} \right)^4 - 2 \left(\frac{N-4}{2} \right)^4 + \left(\frac{N-2}{2} \right)^4 \right) u_{N-3} + \left(\left(\frac{N-4}{2} \right)^4 - \left(\frac{N-2}{2} \right)^4 \right) u_{N-2} \\
 &= \frac{1}{16} \sum_{r=0}^{N-3} \{ (N-2r)^4 - 2(N-2(r+1))^4 + (N-2(r+2))^4 \} u_r \\
 &\quad + \frac{1}{16} ((N-4)^4 - (N-2)^4) u_{N-2}. \tag{8.12}
 \end{aligned}$$

Because we only need to resort to the fourth derivatives $f_0^{(4)}$ for strings satisfying (8.1) but not (8.2), we will exploit the particular structure of these strings in order to simplify this general expression for $f_0^{(4)}$. We commence by considering closely one very specialised type of string : although this may now appear to be a digression, it will ultimately lead us to a superior formula for $f_0^{(4)}$.

Recall from Section 8.2 that for any string of length N which satisfies the inequalities (8.1) but not (8.2), when expressed as a progression of u_r 's, all but $(2h-1)$ of these u_r 's are determined uniquely by (8.6). The remaining u_r 's, namely $u_{m-1}, u_{2m-1}, \dots, u_{(2h-1)m-1}$, each take one of two possible adjacent integer values : typically u_{jm-1} takes either the value ja or $ja+1$, which we described as the lower and higher level of choice for u_{jm-1} .

The $(2h-1)$ of the u_r 's for which there is a choice of value can be matched in pairs as follows :

$$\begin{array}{ll}
 u_{m-1} & \text{with } u_{(2h-1)m-1} \\
 u_{2m-1} & \text{with } u_{(2h-2)m-1} \\
 u_{3m-1} & \text{with } u_{(2h-3)m-1} \\
 & \vdots \\
 u_{(h-1)m-1} & \text{with } u_{(h+1)m-1}
 \end{array}$$

and u_{hm-1} is left over, not in a pair. The particular strings we consider are those for which

one member of each pair takes the higher value and the other the lower value, and we shall call such strings *balanced*. Since, as we saw in Lemma 8.1, h of the $(2h-1)$ u_r 's for which there is a choice of value have to take the higher value, u_{hm-1} is always at the higher level for a balanced string. We now show that all balanced strings of a given length N have the same value for $f_0^{(4)}$.

Proposition 8.2

For a balanced string,

$$f_0^{(4)} = \frac{1}{16} (N^4 - (N-2)^4) + 1. \quad (8.13)$$

Proof

We start by establishing a property, shared by all the strings $s_0 s_1 \dots s_{N-1}$ which satisfy (8.1) and for which $f_0^{(4)} = -N$, relevant to the cancellation of like terms in the sum (8.11). For such a string, we know that u_0, u_1, \dots, u_{m-2} are specified by

$$u_r = \left\lceil \frac{(r+1)a}{m} \right\rceil.$$

Taking the u_r 's in reverse order

$$u_{m-2-r} = \left\lceil \frac{(m-1-r)a}{m} \right\rceil$$

so that

$$u_{m-2-r} = a + \left\lceil -\frac{(r+1)a}{m} \right\rceil = a + \left(1 - \left\lceil \frac{(r+1)a}{m} \right\rceil \right) = a + 1 - u_r. \quad (8.14)$$

Thus for $r = 1, 2, \dots, m-2$,

$$\begin{aligned} t_{m-1-r} &= u_{m-1-r} - u_{m-2-r} \\ &= (a + 1 - u_{r-1}) - (a + 1 - u_r) \\ &= u_r - u_{r-1} = t_r, \end{aligned} \quad (8.15)$$

i.e. the t_r 's have a palindromic character over the range $r = 1$ to $r = m-2$.

Recall from (8.11) the expression for $f_0^{(4)}$ in terms of the t_r 's :

$$f_0^{(4)} = \frac{1}{16} \sum_{r=0}^{N-2} \{ (N-2r)^4 - (N-2r-2)^4 \} t_r.$$

With the exception of the indices $r = 0; m-1, m; 2m-1, 2m; 3m-1, 3m; \dots; (2h-1)m-1, (2h-1)m$,

$$t_{N-1-r} = t_r$$

because of the palindromic character of the t_r 's in the range 1 to $m-2$ just mentioned, and because of their periodic repetition between the ranges 1 to $m-2$, $m+1$ to $2m-2$, ..., $(2h-1)m+1$ to $2hm-2 = N-2$. But

$$\begin{aligned} N-2(N-1-r) &= -(N-2r-2), \\ \text{and } N-2(N-1-r)-2 &= -(N-2r), \end{aligned}$$

so the majority of terms in the sum for $f_0^{(4)}$ cancel; the indices r other than those excepted make no contribution to the sum.

Now for a balanced string and for each $1 \leq j \leq h-1$ either statement (a) or statement (b), as follows, holds :

(a) u_{jm-1} takes its higher value and $u_{N-2-(jm-1)}$ takes its lower value. In this case

$$t_{jm-1} = 1, t_{jm} = 0 \text{ and } t_{N-2-(jm-1)} = 0, t_{N-1-(jm-1)} = 1.$$

(b) u_{jm-1} takes its lower value and $u_{N-2-(jm-1)}$ takes its higher value. In this case

$$t_{jm-1} = 0, t_{jm} = 1 \text{ and } t_{N-2-(jm-1)} = 1, t_{N-1-(jm-1)} = 0.$$

But both possibilities imply $t_r = t_{N-1-r}$ for the two indices $r = jm-1$ and $r = jm$. This means we only get a contribution to the sum for $f_0^{(4)}$ from the two indices $r = 0$ and $r = hm-1 = \frac{N}{2}-1$, and so

$$\begin{aligned} f_0^{(4)} &= \frac{1}{16} \{ N^4 - (N-2)^4 \} + \frac{1}{16} \left\{ \left(N-2\left(\frac{N}{2}-1\right) \right)^4 - \left(N-2\left(\frac{N}{2}-1\right)-2 \right)^4 \right\} \\ &= \frac{1}{16} (N^4 - (N-2)^4) + 1. \end{aligned}$$

□

Corollary 8.3

A balanced string does not generate a periodic sequence admissible adjacent to zero.

Proof

For a balanced string,

$$\begin{aligned} (-g)^{(4)} - f_0^{(4)} &= 2N \left(\frac{N^2}{4} + 1 \right) - \frac{1}{16} (N^4 - (N-2)^4) - 1 \\ &= 2N \left(\frac{N^2}{4} + 1 \right) - \frac{1}{16} (8N^3 - 24N^2 + 32N - 16) - 1 \\ &= \frac{3}{2}N^2 > 0. \end{aligned}$$

□

We now resume the investigation of an arbitrary string which satisfies inequalities (8.1) but not (8.2). Let $s_0s_1 \dots s_{N-1}$ denote such a string, and suppose it is adjusted as previously (by cycling and negating where necessary) so that we may suppose $f_0'' = -N$. Let u_0, u_1, \dots, u_{N-1} be the associated progression of u_r 's. Apart from the $(2h-1)$ u_r 's for which a choice of values is available, the u_r 's associated to the string $s_0s_1 \dots s_{N-1}$ are precisely the same as the u_r 's corresponding to a balanced string. Thus in the sum for $f_0^{(4)}$ (from (8.12) above),

$$\begin{aligned} f_0^{(4)} &= \frac{1}{16} \sum_{r=0}^{N-3} \{ (N-2r)^4 - 2(N-2(r+1))^4 + (N-2(r+2))^4 \} u_r \\ &\quad + \frac{1}{16} ((N-4)^4 - (N-2)^4) u_{N-2}, \quad (8.16) \end{aligned}$$

all the terms, except for the $(2h-1)$ noted above, will contribute the same amount as for a balanced string. Denoting the value of $f_0^{(4)}$ for a balanced string by $\overline{f_0^{(4)}}$, this means that $f_0^{(4)} - \overline{f_0^{(4)}}$ can be written as a sum with only $(2h-1)$ terms. Our explicit formula, in Proposition 8.2, for $\overline{f_0^{(4)}}$ in terms of N can then be substituted, to give a formula for $f_0^{(4)}$, derived from $s_0s_1 \dots s_{N-1}$ involving just $(2h-1)$ summands.

Define, for $1 \leq j \leq 2h-1$, $e_j = u_{jm-1} - ja$, so that

$$e_j = \begin{cases} 0 & \text{if } u_{jm-1} \text{ takes its lower value,} \\ 1 & \text{if } u_{jm-1} \text{ takes its higher value.} \end{cases}$$

Then, by comparing the terms $r = m-1, 2m-1, \dots, (2h-1)m-1$ in formula (8.16) for both a balanced string and our general string, we find

$$f_0^{(4)} - \overline{f_0^{(4)}} = \sum_{j=1}^{h-1} A_j (e_j + e_{2h-j} - 1) + 2(e_h - 1) \quad (8.17)$$

where $A_j = \frac{1}{16} \{ (N - 2jm + 2)^4 - 2(N - 2jm)^4 + (N - 2jm - 2)^4 \}$.

We now simplify by using the identity $(x+a)^4 - 2x^4 + (x-a)^4 = 12a^2x^2 + 2a^4$:

$$\begin{aligned} f_0^{(4)} - \overline{f_0^{(4)}} &= \sum_{j=1}^{h-1} (3(N - 2jm)^2 + 2) (e_j + e_{2h-j} - 1) + 2(e_h - 1) \\ &= 12m^2 \sum_{j=1}^{h-1} (h-j)^2 (e_j + e_{2h-j} - 1) \\ &\quad + 2 \sum_{j=1}^{h-1} (e_j + e_{2h-j} - 1) + 2(e_h - 1) \\ &= 12m^2 \sum_{j=1}^{h-1} (h-j)^2 (e_j + e_{2h-j} - 1), \end{aligned} \quad (8.18)$$

where the final simplification uses $e_1 + e_2 + \dots + e_{2h-1} = h$, which is true because precisely h of the u_r 's, where there is a choice of values, must be set at the higher level. Notice that the quantity m^2 , which relates to the length of the string, is not present within the summation. This will be an important aspect of the formula in the applications to follow.

Example 8.2 : Calculation of $f_0^{(4)}$ when $N = 42$ and $x = 18$

We return to Example 8.1, involving the 10 strings for $N = 42$ and $x = 18$, and use the formula (8.18) to calculate $f_0^{(4)}$ for each of these strings. Note that the scheme of L's and H's introduced earlier corresponds precisely to the quantities $e_1, e_2, \dots, e_{2h-1}$ if L corresponds to a 0 and an H to a 1. For string number 8 in the list, with scheme of choices

(L) H H L L H we have

$$e_1 = 1, \quad e_2 = 1, \quad e_3 = 0, \quad e_4 = 0, \quad e_5 = 1,$$

so that

$$\begin{aligned} f_0^{(4)} &= \overline{f_0^{(4)}} + 12 \cdot 7^2 \sum_{j=1}^2 (3-j)^2 (e_j + e_{6-j} - 1) \\ &= \frac{1}{16} (42^4 - 40^4) + 1 + 12 \cdot 7^2 \cdot \{2^2(e_1 + e_5 - 1) + 1^2(e_2 + e_4 - 1)\} \\ &= 36834. \end{aligned}$$

In the same way we can compute $f_0^{(4)}$ for each of the 10 strings, and these values are listed in the following table.

| Scheme of Choices | | | | | | | $f_0^{(4)}$ | |
|-------------------|-----|---|---|---|---|---|-------------|------------|
| 1. | (L) | L | L | H | H | H | 34482 | (balanced) |
| 2. | (L) | L | H | L | H | H | 35070 | |
| 3. | (L) | L | H | H | L | H | 34482 | (balanced) |
| 4. | (L) | L | H | H | H | L | 32718 | |
| 5. | (L) | H | L | L | H | H | 36834 | |
| 6. | (L) | H | L | H | L | H | 36246 | |
| 7. | (L) | H | L | H | H | L | 34482 | (balanced) |
| 8. | (L) | H | H | L | L | H | 36834 | |
| 9. | (L) | H | H | L | H | L | 35070 | |
| 10. | (L) | H | H | H | L | L | 34482 | (balanced) |

In this case $(-g)^{(4)} = 37128$, so we see that none of the strings generates a periodic sequence, as anticipated.

8.4 A suggested strategy for completing the argument

It is tempting to expect that Example 8.2 above is typical, and that any string which satisfies the inequalities (8.1) with $f_0'' = -N$ will have $f_0^{(4)} < (-g)^{(4)}$. Unfortunately this is not true, as our next example shows.

Example 8.3 : $f_0^{(4)}$ can exceed $(-g)^{(4)}$

$$\left. \begin{array}{l} N = 16 = 2 \times 2 \times 4 \\ x = 8 = 2 \times 1 \times 4 \end{array} \right\} \text{so } h = 4, m = 2 \text{ and } a = 1.$$

There are $\binom{2h-1}{h} = 35$ possible strings to consider for this specific choice of N and x , because $2h - 1 = 7$ of the u_r 's are not uniquely determined by (8.5). Each string must have four of these u_r 's set to the higher value and three set to the lower value. We consider just one of the strings, that with the scheme of choices (L) H H L L L H H. Our formulae for $(-g)^{(4)}$, $\overline{f_0^{(4)}}$ and $f_0^{(4)}$ give

$$f_0^{(4)} = 2272 \quad \text{and} \quad (-g)^{(4)} = 2080,$$

so for this string $f_0^{(4)} \not\leq (-g)^{(4)}$.

Despite this, the string exhibited in this example *does not* generate a periodic sequence. Although this string may not be ruled out on the basis of $f_0^{(4)}$, it could be that one of the other fourth derivatives $f_r^{(4)}$ can be used instead, or equivalently $f_0^{(4)}$ for a cycled version of the string. As we saw earlier, there are $h - 1 = 3$ cycled versions which also have their $f_0'' = -N$. The corresponding fourth derivatives $f_0^{(4)}$ for all four schemes with $f_0'' = -N$ are given in the table below.

| | | | | | | | | $f_0^{(4)}$ |
|-----|---|---|---|---|---|---|---|-------------|
| (L) | H | H | L | L | L | H | H | 2272 |
| (L) | L | L | H | H | L | H | H | 1696 |
| (L) | L | H | H | L | H | H | L | 1504 |
| (L) | H | H | L | H | H | L | L | 1696 |

Evidently the value of $f_0^{(4)}$ for all three of the cycled versions is well below $(-g)^{(4)}$, and so this string does not generate a periodic sequence. Clearly then a general argument, based on comparing $f_0^{(4)}$ with $(-g)^{(4)}$, for ruling out the strings we are considering, must necessarily involve not only the strings themselves but also certain of their cycled states.

Note that for the string in Example 8.3, the cycled state which gives rise to the largest

value of $f_0^{(4)}$ is the one where the H choices are biased towards the two ends of the scheme. This comes as no surprise because the places at the two ends of the scheme correspond to the term with the largest coefficients $(h-j)^2$ in the sum (8.18) for $f_0^{(4)} - \overline{f_0^{(4)}}$. Considerations of this sort suggest that there will always be a cycled state for which $f_0^{(4)} < (-g)^{(4)}$. A balanced string, which has the H's and L's distributed evenly between the places near the ends of the scheme (which contribute a large amount to the sum) and the places near the centre (which contribute little to the sum), definitely has $f_0^{(4)} < (-g)^{(4)}$ as we saw in Corollary 8.3. In view of this it is difficult to conceive of the existence of a string with $f_0^{(4)} > (-g)^{(4)}$ for each of its h possible cycled versions, because then the H's would need to be biased towards the ends of the scheme in all these cycled states.

The idea promoted in the previous paragraph depends crucially on the phrase "biased towards the ends of the scheme", for which it seems hard to provide a precise interpretation. Instead we concentrate on one special string which arises naturally when we consider the process of cycling. The string is one with the scheme of choices (L) H L H ... L H, so that the $(2h-1)$ u_r 's at which there is a choice of value alternately assume the lower and the higher level. The crucial feature of this string is that it is invariant when we cycle (but of course maintain $f_0'' = -N$), and so the value of $f_0^{(4)}$ does not change.

Lemma 8.4

For the string with the alternating scheme of choices (L) H L H ... L H,

$$f_0^{(4)} = \overline{f_0^{(4)}} + 12m^2 \left(\frac{1}{2} h(h-1) \right). \quad (8.19)$$

Proof

$$\begin{aligned} f_0^{(4)} &= \overline{f_0^{(4)}} + 12m^2 \sum_{j=1}^{h-1} (h-j)^2 (e_j + e_{(2h-j)} - 1) \\ &= \overline{f_0^{(4)}} + 12m^2 \sum_{j=1}^{h-1} (-1)^{j-1} (h-j)^2 \\ &= \overline{f_0^{(4)}} + 12m^2 (-1)^h \sum_{k=1}^{h-1} (-1)^{k-1} k^2 \end{aligned}$$

$$\text{and } \sum_{k=1}^{h-1} (-1)^{k-1} k^2 = \begin{cases} -\frac{1}{2}h(h-1) & \text{if } h \text{ is odd,} \\ \frac{1}{2}h(h-1) & \text{if } h \text{ is even.} \end{cases}$$

□

Thus the value $f_{0[\text{alt}]}^{(4)}$ of $f_0^{(4)}$ for an alternating string is larger than for a balanced string; even so, an alternating string does not generate a periodic sequence admissible adjacent to zero, because

$$\begin{aligned} (-g)^{(4)} - f_{0[\text{alt}]}^{(4)} &= (-g)^{(4)} - \overline{f_0^{(4)}} - 12m^2 \left(\frac{1}{2}h(h-1) \right) \\ &= \frac{3}{2}N^2 - 6m^2h(h-1) \\ &= 6m^2h^2 - 6m^2h(h-1) = 6m^2h > 0. \end{aligned}$$

It would seem likely that amongst the $\binom{2h-1}{h}$ strings we need to consider here, it is the alternating string which provides the “worst” value of $f_0^{(4)}$, in the following sense.

Conjecture 8.5

If $s_0s_1 \dots s_{N-1}$ is any of the $\binom{2h-1}{h}$ strings which satisfies inequalities (8.1) but not (8.2) then

- (a) there is a cycled state of $s_0s_1 \dots s_{N-1}$ with $f_0'' = -N$ and for which $f_0^{(4)} \leq f_{0[\text{alt}]}^{(4)}$;
- (b) the average of the values of $f_0^{(4)}$, over the h cycled states of $s_0s_1 \dots s_{N-1}$ with $f_0'' = -N$, does not exceed $f_{0[\text{alt}]}^{(4)}$.

Some remarks on the conjecture

1. Establishing *either* (a) *or* (b) would mean that the inequalities (8.1) could be strengthened to (8.2) for all even N , and this would complete the task of classifying the strings of even length which generate periodic sequences admissible adjacent to zero. Note that statement (b) is stronger than statement (a) because if the average of $f_0^{(4)}$ for all h cycled states is less than $f_{0[\text{alt}]}^{(4)}$ then at least one of these cycled states must provide a value of $f_0^{(4)}$ less than $f_{0[\text{alt}]}^{(4)}$.

2. The formula (8.18) for $f_0^{(4)}$ can be used to simplify the condition that $f_0^{(4)} \leq f_{0[\text{alt}]}^{(4)}$. For any string $s_0 s_1 \dots s_{N-1}$ with $f_0'' = -N$ which satisfies inequalities (8.1) but not (8.2),

$$\begin{aligned} f_{0[\text{alt}]}^{(4)} - f_0^{(4)} &= \left(\overline{f_0^{(4)}} + 12m^2 \sum_{j=1}^{h-1} (h-j)^2 (\tilde{e}_j + \tilde{e}_{2h-j} - 1) \right) \\ &\quad - \left(\overline{f_0^{(4)}} + 12m^2 \sum_{j=1}^{h-1} (h-j)^2 (e_j + e_{2h-j} - 1) \right) \\ &= 12m^2 \left(\sum_{j=1}^{h-1} (h-j)^2 (\tilde{e}_j + \tilde{e}_{2h-j}) - \sum_{j=1}^{h-1} (h-j)^2 (e_j + e_{2h-j}) \right) \end{aligned}$$

where \tilde{e}_j stands for e_j in the case of the alternating string. It follows that

$$f_{0[\text{alt}]}^{(4)} - f_0^{(4)} \geq 0 \iff \sum_{j=1}^{h-1} (h-j)^2 (\tilde{e}_j + \tilde{e}_{2h-j}) \geq \sum_{j=1}^{h-1} (h-j)^2 (e_j + e_{2h-j}). \quad (8.20)$$

If now we substitute the formula for the value of $f_{0[\text{alt}]}^{(4)}$ from Lemma 8.4,

$$\sum_{j=1}^{h-1} (h-j)^2 (\tilde{e}_j + \tilde{e}_{2h-j}) = \sum_{j=1}^{h-1} (h-j)^2 + \frac{1}{2}h(h-1) = \frac{1}{3}h(h^2-1),$$

so

$$f_{0[\text{alt}]}^{(4)} - f_0^{(4)} \geq 0 \iff \frac{1}{3}h(h^2-1) \geq \sum_{j=1}^{h-1} (h-j)^2 (e_j + e_{2h-j}). \quad (8.21)$$

The notable feature of the conditions (8.20) and (8.21) is that the quantity m is no longer involved. Thus whether a cycled version of a string (or the average of all h cycled versions) gives a value of $f_0^{(4)}$ less than $f_{0[\text{alt}]}^{(4)}$ does not depend on the length of the string, but only on h , the number of u_r 's for which a choice of values is possible (and originally $h = \text{hcf}(\frac{1}{2}x, \frac{1}{2}N)$).

For example, if we could establish the conjecture when there were, say, 5 u_r 's with a choice of value (and we can, see below) then no string which satisfied (8.1) but failed to satisfy (8.2) and for which $\text{hcf}(\frac{1}{2}x, \frac{1}{2}N) = 3$ could generate a periodic sequence admissible adjacent to zero, irrespective of the actual string length.

| h | Scheme of Choices | | | | | | S_{\min} | S_{av} | $\frac{1}{3}h(h^2 - 1)$ |
|-----|-------------------|---|---|---|---|---|------------|-----------------|-------------------------|
| 1 | (L) | H | | | | | 0 | 0 | 0 |
| 2 | (L) | L | H | H | | | 1 | 1 | 2 |
| | (L) | H | L | H | | | 2 | 2 | 2 |
| | (L) | H | H | L | | | 1 | 1 | 2 |
| 3 | (L) | L | L | H | H | H | 2 | 4 | 8 |
| | (L) | L | H | L | H | H | 5 | $6\frac{2}{3}$ | 8 |
| | (L) | L | H | H | L | H | 5 | $6\frac{2}{3}$ | 8 |
| | (L) | L | H | H | H | L | 2 | 4 | 8 |
| | (L) | H | L | L | H | H | 5 | $6\frac{2}{3}$ | 8 |
| | (L) | H | L | H | L | H | 8 | 8 | 8 |
| | (L) | H | L | H | H | L | 5 | $6\frac{2}{3}$ | 8 |
| | (L) | H | H | L | L | H | 5 | $6\frac{2}{3}$ | 8 |
| | (L) | H | H | L | H | L | 5 | $6\frac{2}{3}$ | 8 |
| | (L) | H | H | H | L | L | 2 | 4 | 8 |

Table 8.1: Verification of the conjecture for $h = 1, 2$ and 3 by listing all cases.

3. When h is small, the number $\binom{2h-1}{h}$ of schemes of choices is small enough for all the possibilities to be listed. For a particular scheme of $(2h - 1)$ choices, define the following quantities : S_{\min} is the minimum value of the sum $\sum_{j=1}^{h-1} (h-j)^2 (e_j + e_{2h-j})$ and S_{av} is the average value of the sum, taken in each case over the h cycled versions of the scheme,

$$S_{\min} = \min_{\text{cycle}} \sum_{j=1}^{h-1} (h-j)^2 (e_j + e_{2h-j}) \quad (8.22)$$

$$S_{\text{av}} = \frac{1}{h} \sum_{\text{cycle}} \sum_{j=1}^{h-1} (h-j)^2 (e_j + e_{2h-j}). \quad (8.23)$$

The values of S_{\min} and S_{av} are listed in Table 8.1 for $h = 1, 2$ and 3 and for all schemes of choices. Both S_{\min} and S_{av} are below $\frac{1}{3}h(h^2 - 1)$, in accordance with the conjecture.

We have confirmed our conjecture for all values $h \leq 15$, by calculating the value of S_{av} for each of the $\binom{2h-1}{h}$ schemes of L's and H's.

Although we have not been able to establish that every string which satisfies (8.1) but fails to satisfy (8.2) has some cycled state for which $f_0^{(4)} = -N$ and $f_0^{(4)} < (-g)^{(4)}$, the outstanding task has been reduced to a definite combinatorial problem which can be explored independently of the context of admissible sequences and their generating strings, where it originated. The combinatorial problem has been affirmatively solved up to $h = 15$, by exhaustive checking on computer.

Even the modest achievement of establishing the conjecture for small h values has a substantial payoff in terms of the classification of generating strings when N is even. This is the subject of the next section, where we show that on the basis of our partial resolution of the conjecture, a precise classification can be given for an infinity of even N values.

8.5 Even length strings for which a precise classification can be given

In the last section we studied in detail those strings which satisfy the inequalities (8.1) but fail to satisfy their strengthened version (8.2). Ideally we would like to have shown that all such strings do not generate a periodic sequence, and we attempted this by considering the fourth derivatives. Although the general result was not established, the desired conclusion was obtained in certain cases. Here we start by listing these cases.

Recall that for the task of classification, if we suppose $s_0 s_1 \dots s_{N-1}$ to be an even length string which satisfies (8.1) but not (8.2) then, after cycling and negation as necessary, we may assume $f_0'' = -N$. This done, we calculate the progression of u_r 's, and set $x = u_{N-1}$. The string does not generate a periodic sequence admissible adjacent to zero if the integers x and N satisfy either (i) x is odd, or (ii) x is even and $h = \text{hcf}(\frac{1}{2}x, \frac{1}{2}N) \leq 15$.

A difficulty present for even N is that we *do not know* that a string which generates a periodic sequence admissible adjacent to zero consists of runs of 0's and runs of +- pairs. This was a central piece of information in the case when N is odd, because it enabled us to give a separate meaning to x , intrinsic to the string under consideration, and independent of

the cycled state in which it was written. Unfortunately the results that enabled us to restrict the format of the strings to runs of 0's and $+ -$ pairs, Propositions 7.1 and 7.2 from Section 7.3, depended upon the strengthened inequalities for their proof. And even though we know that strings satisfying either (i) or (ii) above cannot generate periodic sequences admissible adjacent to zero, these statements are of little significance in the absence of some intrinsic description of x , independent of any particular cycled state of the string.

Clearly then, some extra information is necessary to make any progress with the classification of the even length strings that generate periodic sequences admissible adjacent to zero. For this we turn now to the earlier results proved in Chapter 3 about the family of strings $+0 \dots 0+$. We were able to describe there the precise ranges of θ values for which such a string can appear within some admissible sequence. The specific consequence that we need here is that the string $+0 \dots 0+$ of length n cannot appear as a substring of any admissible sequence when θ is less than $\frac{\pi}{n}$. And this means that the string $+0 \dots 0+$ cannot appear as a substring of a string $s_0 s_1 \dots s_{N-1}$ generating a periodic sequence admissible in an interval with zero as left end-point.

Thus we can establish Proposition (7.1) by an independent method, without recourse to the strengthened inequalities (8.2). This will be valuable extra information permitting an interpretation of x , independent of the definition $x = u_{N-1}$ which refers to a particular cycled state of the string, and enabling us to proceed with the classification.

Proposition 8.6 (Precise classification when N is twice a prime)

When N is twice an odd prime, the classification of Section 8.1 accounts for all strings $s_0 s_1 \dots s_{N-1}$ that generate periodic sequences admissible adjacent to zero.

Proof

Suppose that $s_0 s_1 \dots s_{N-1}$ is a string that generates a periodic sequence admissible adjacent to zero, but does not satisfy the strengthened inequalities (8.2). In Section 8.2 we saw that such a string must satisfy the inequalities (8.1), and that (after cycling and negation) $f_0'' = -N$. Determine u_0, u_1, \dots, u_{N-1} , from the s_r 's via the t_r 's, and let $x = u_{N-1}$; then again from Section 8.2 we know that x must assume one of the values $0, 2, 4, \dots, N$.

Because the string $s_0 s_1 \dots s_{N-1}$ cannot contain either $+0\dots 0+$ or $-0\dots 0-$ as a sub-string (and this prohibition includes $++$ and $--$), the *non-zero* s_r 's within the string are alternately $+$ and $-$. The number of $+$'s and $-$'s must balance because $s_0 + s_1 + \dots + s_{N-1} = 0$. Thus, allowing for wrap-around, the string consists of alternating $+0\dots 0$ and $-0\dots 0$ runs, so has the form

$$0 \dots 0 + 0 \dots 0 - 0 \dots 0 + 0 \dots 0 - 0 \dots 0 \dots + 0 \dots 0 - 0 \dots 0$$

or

$$0 \dots 0 - 0 \dots 0 + 0 \dots 0 - 0 \dots 0 + 0 \dots 0 \dots - 0 \dots 0 + 0 \dots 0$$

with the understanding that any of the runs of zeros may be absent. The corresponding t_r 's for these two types of string are as follows.

$$\begin{array}{ll} s_r & 0 \dots 0 + 0 \dots 0 - 0 \dots 0 + 0 \dots 0 - 0 \dots 0 \dots + 0 \dots 0 - 0 \dots 0 \\ t_r & 0 \dots 0 \ 1 \ 1 \dots 1 \ 0 \ 0 \dots 0 \ 1 \ 1 \dots 1 \ 0 \ 0 \dots 0 \dots 1 \ 1 \dots 1 \ 0 \ 0 \dots 0 \end{array}$$

or

$$\begin{array}{ll} s_r & 0 \dots 0 - 0 \dots 0 + 0 \dots 0 - 0 \dots 0 + 0 \dots 0 \dots - 0 \dots 0 + 0 \dots 0 \\ t_r & 0 \dots 0 \ -1 \ -1 \dots -1 \ 0 \ 0 \dots 0 \ -1 \ -1 \dots -1 \ 0 \ 0 \dots 0 \dots -1 \ -1 \dots -1 \ 0 \ 0 \dots 0 \end{array}$$

Since $x = u_{N-1}$ is the sum of all the t_r 's, our string fits with the first of these two types because we know x to be positive. There must be at least one $+$ in the string, and consequently $x > 0$. Similarly there must be at least one $-$ (to balance the $+$), so $x < N$.

Having ruled out the possibilities $x = 0$ and $x = N$, the results in Section 8.2 can be applied, and these imply that all but $(2h - 1)$ of the u_r 's are uniquely determined. But $h = \text{hcf}(\frac{1}{2}N, \frac{1}{2}x) = 1$, and we found that when $h = 1$ there is only one string satisfying inequalities (8.1) but not (8.2), and that because of fourth derivative considerations this string does not generate a periodic sequence admissible adjacent to zero. Thus we have contradicted our assumption about $s_0 s_1 \dots s_{N-1}$. \square

We could handle the case when N is twice a prime because $\frac{1}{2}x$ and $\frac{1}{2}N$ have highest common factor 1. For the general case when N is even, in the absence of a proof of our

conjecture, the best we can achieve is a partial classification of the strings which generate periodic sequences admissible adjacent to zero. Values of x can arise for which the highest common factor of $\frac{1}{2}x$ and $\frac{1}{2}N$ exceeds 15, and then we cannot rule out the existence of strings which generate periodic sequences admissible adjacent to zero, yet do not satisfy the strengthened inequalities (8.2). Note that, in seeking a classification for certain values of x only, we do now need to show that $x = u_{N-1}$ maintains the same value when a string is cycled : it is no longer possible to sidestep this issue as we could when N was twice a prime.

As above, let $s_0s_1 \dots s_{N-1}$ be a string that generates a periodic sequence admissible adjacent to zero but does not satisfy the strengthened inequalities (8.2). In the proof of Proposition 8.6 we ruled out the cases $x = 0$ and $x = N$ on the basis that $+0 \dots 0+$ cannot appear as a substring of $s_0s_1 \dots s_{N-1}$. With a similar argument, but being more careful with the book-keeping, we can limit the integer x to the range $1 \leq x \leq \frac{1}{2}N$.

Since $+0 \dots 0+$ cannot appear as its substring, the string $s_0s_1 \dots s_{N-1}$ must consist of alternating $+0 \dots 0$ and $-0 \dots 0$ runs. The sum of the lengths of all the $+0 \dots 0$ runs plus the sum of the lengths of all the $-0 \dots 0$ runs equals the length N of the string (allowing for wrap-around). By negating $s_0s_1 \dots s_{N-1}$ if necessary we obtain a string where the sum of the lengths of all the $+0 \dots 0$ runs does not exceed the sum of the lengths of all the $-0 \dots 0$ runs, so is at most $\frac{1}{2}N$. Whether or not we have negated, we can cycle the string until $f_0'' = -N$; this for the reason that at the locations where u_r is eligible for higher or lower level values, then $f_r'' = \pm N$ and there have to be these locations for either possibility. In this cycled state, the first digit of the string must be a $+$ by the formula $u_0 = \lceil \frac{x}{N} \rceil = 1$ (regardless of x) and $s_0 = t_0 = u_0 = 1$. Thus the string can be written as

$$+ 0 \dots 0 - 0 \dots 0 + 0 \dots 0 - 0 \dots 0 \dots + 0 \dots 0 - 0 \dots 0$$

and $x = u_{N-1}$ is the sum of the lengths of all the $+0 \dots 0$ runs, and $x \leq \frac{1}{2}N$.

When N is twice an odd number we can actually go one step further and say that $x < \frac{1}{2}N$, in which case we can recover the familiar situation where we know the string consists of runs of 0's and runs of $+-$ pairs.

Lemma 8.7

When N is twice an odd number, a string $s_0 s_1 \dots s_{N-1}$ which satisfies the inequalities (8.1), but not their strengthened form (8.2), consists of runs of $+-$ pairs and runs of zeros, and $x = u_{N-1}$ is the count of the number of $+-$ pairs in the string.

Proof

There are no strings with $x = \frac{1}{2}N$ which satisfy (8.1) but not (8.2) because $\frac{1}{2}N$ is odd; thus $1 \leq x < \frac{1}{2}N$. Now since $s_0 s_1 \dots s_{N-1}$ satisfies (8.1), we have

$$-\frac{1}{2}N \leq y - \left(\frac{1}{2}(N-1) - r\right)x - Nu_r \leq \frac{1}{2}N \quad \text{for } r = 0, 1, \dots, N-1$$

with $x = u_{N-1}$, $y = u_0 + u_1 + \dots + u_{N-1}$. Rearranging the left-hand inequality we find

$$Nu_r \leq y - \left(\frac{1}{2}(N-1) - r\right)x + \frac{1}{2}N \quad (8.24)$$

and using the right-hand inequality with $r = N-1$,

$$y - \left(\frac{1}{2}(N-1) - (N-1)\right)x - Nx \leq \frac{1}{2}N. \quad (8.25)$$

Taken together,

$$u_r \leq \frac{(r+1)x}{N} + 1 \quad (8.26)$$

and since $1 \leq x < \frac{N}{2}$, it follows that

$$u_r < \frac{r+1}{2} + 1 = \frac{r+3}{2}. \quad (8.27)$$

The string $s_0 s_1 \dots s_{N-1}$ (appropriately cycled) starts with $s_0 = +$, and we know that $s_1 = 0$ or $-$. We now see that $s_1 = 0$ is impossible, because otherwise $t_0 = t_1 = 1$ and $u_1 = 2$ which contradicts the inequality $u_r < \frac{1}{2}(r+3)$, just obtained. Consequently the string $s_0 s_1 \dots s_{N-1}$ commences with a $+-$ pair. If this is followed by a run of zeros, cycle until the string now starts with the next $+$; $x = u_{N-1}$ does not change, so as before $+$ is followed immediately by $-$, and continue in this way. \square

Our knowledge about the case when N is twice an odd number is summarised in the next proposition, whose proof is essentially the previous discussion.

Proposition 8.8 (Partial classification when N is twice an odd number)

When N is twice an odd number, a string of length N which generate a periodic sequence admissible adjacent to zero, consists of runs of $+-$ pairs and runs of zeros. The integer x , denoting the number of $+-$ pairs in the string, may be freely specified between 1 and $\frac{1}{2}N$. If either (i) x is odd, or (ii) $\text{hcf}(\frac{1}{2}x, \frac{1}{2}N) \leq 15$, then there is a unique string (up to cycling and negation) with x $+-$ pairs which generates a periodic sequence admissible adjacent to zero, and this string is given within the classification described in Section 8.1. For other values of x , the string produced by the construction of Section 8.1 does generate a periodic sequence admissible adjacent to zero, but we cannot rule out the existence of additional strings with x $+-$ pairs that generate periodic sequences admissible adjacent to zero.

The remaining case, when N is twice an even number, is more problematic, because we have no simple way of excluding $x = \frac{1}{2}N$, and Lemma 8.7 does not apply. Thus we are forced to work with the identification of x as the sum of the lengths of all the $+0 \dots 0$ runs. But despite being a less convenient interpretation for x , it is still a legitimate way to assign the strings of length N to different classes, just as we did previously using the count of $+-$ pairs as our parameter. Once again, if x is odd, or $\text{hcf}(\frac{1}{2}x, \frac{1}{2}N) \leq 15$, there is a unique string with that value of x ; for other x we have been unable to rule out exotic strings in addition to the familiar ones identified in the classification of Section 8.1.

To exclude the case $x = \frac{1}{2}N$, it seems as though we need extra information of the sort provided by Proposition 7.2, namely that the substrings $0+-+ \dots +-+0$ cannot appear within a string which generates a periodic sequence admissible adjacent to zero. It was noted in Chapter 3 that the methods developed there have a good chance of extending to the strings $0+-+ \dots +-+0$ and that the ranges of θ can be described where they appear as substrings of admissible sequences and this would provide an independent means of establishing Proposition 7.2. In this way, x could once again be identified with the number of $+-$ pairs, and the case when N is twice an even number treated in the same manner as when N is twice an odd number.

Chapter Summary

- Each string of even length arising via the formula $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$, as described in Chapter 7, generates a periodic sequence admissible adjacent to zero. The computer search results suggest that these and their cycled versions account for all the even period periodic sequences admissible adjacent to zero.
- The parity argument used in Chapter 7 to effect a key strengthening of the differential form of the Chua-Lin inequalities does not now apply. To demonstrate the presumed classification it is necessary to eliminate strings for which one or more of the second derivatives f_r'' coincides with g'' or $-g''$ when evaluated at $\theta = 0$.
- The strings of even length N to be ruled out are those derived from a progression of u_r 's satisfying $\left\lceil \frac{(r+1)a}{m} \right\rceil \leq u_r \leq \left\lfloor \frac{(r+1)a}{m} \right\rfloor + 1$ for $r = 0, 1, \dots, N-2$ and $u_{N-1} = x$, where $N = 2mh$, $x = 2ah$, $h = \text{hcf}(\frac{1}{2}x, \frac{1}{2}N)$, subject to the constraint that at the $(2h-1)$ positions where there is a choice of two consecutive integers for u_r , precisely h must be set at the higher level.
- The task of excluding these surplus strings is reduced, via consideration of higher derivatives, to the resolution of a specific combinatorial problem, dependent upon h but not N or x . An affirmative solution is verified computationally for small values of h ; we conjecture its universal validity, and consequently that none of the supernumerary strings generates a periodic sequence admissible adjacent to zero.
- The partial resolution of this combinatorial problem, and information from Chapter 3 specifying when a substring $+0 \dots 0+$ can appear within some admissible sequence, permits the exclusion of the surplus strings if the string length N is twice a prime, or if x is odd, or if $h \leq 15$. The classification of Chapter 7 via $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$, effective for odd period, then accounts entirely for these even period periodic sequences admissible adjacent to zero.

Chapter 9

Periodic Sequences Admissible in an Interval Having Zero as Left End-point : Structure of the Generating Strings and Size of the Intervals of Admissibility

We present several topics to do with periodic sequences admissible adjacent to zero, quite distinct from the task of classifying them. The first concerns the structure of the generating strings, more particularly the compatibility of the lengths of the runs of zeros, and of the runs of $+-$ pairs. For the next we consider the specific character of the trigonometric sum functions appearing in the Chua-Lin inequalities. Certain functions of a simple form are always present and we review whether this fact might afford a separate characterisation of the periodic sequences that are admissible adjacent to zero. Consideration of the dominant role assumed by these functions amongst the full set leads to a theoretical confirmation of our observation, from the computer search results, that all the sequences with generating string of length N have interval of admissibility $(0, \frac{1}{N})$. The final section investigates the orbits in state-space associated with the periodic sequences admissible adjacent to zero; we obtain thereby a new lower bound for the proportion of the state-space taken up by the points which give rise to periodic sequences.

9.1 Remarks on the structure of strings generating periodic sequences admissible adjacent to zero

(To avoid complicating the presentation we assume in the following that N is odd.)

We take as starting point the results of Sections 7.1 to 7.3. To resume from there, if a string $s_0s_1 \dots s_{N-1}$ generates a periodic sequence admissible adjacent to zero, then the corresponding u_r 's (which arise from the s_r 's via the t_r 's) satisfy the inequalities

$$-\frac{1}{2}(N-1) \leq y - \left(\frac{1}{2}(N-1) - r\right)x - Nu_r \leq \frac{1}{2}(N-1) \quad \text{for } r = 0, 1, \dots, N-1 \quad (9.1)$$

where $x = u_{N-2} = u_{N-1}$ and $y = u_0 + u_1 + \dots + u_{N-1}$. We know also that the string $s_0s_1 \dots s_{N-1}$ consists of runs of zeros and runs of $+-$ pairs.

Our first result relates to the possible lengths of runs of zeros that can be present, and essentially shows that the zeros are distributed evenly between the runs of $+-$ pairs.

Proposition 9.1 (Lengths of runs of zeros)

Suppose $s_0s_1 \dots s_{N-1}$ is a string that generates a periodic sequence admissible adjacent to zero. The runs of zeros within the string $s_0s_1 \dots s_{N-1}$ (including any run present because of wrap-around) are of at most two lengths, and these lengths differ by 1.

This proposition can be proved by elaborating the techniques employed in the proofs of Propositions 7.1 and 7.2 in Section 7.3 (i.e. by finding the maximum of some linear function of x and y when these variables are constrained to an appropriate parallelogram in the xy -plane). It is of more interest, though, to illustrate an alternative strategy. Our proof will employ a general property of the u_r 's, namely that the progression of the u_r 's derived from a string $s_0s_1 \dots s_{N-1}$ that generates a periodic sequence admissible adjacent to zero, is subadditive.

Lemma 9.2 (Subadditivity of the u_r 's)

The progression of integers u_0, u_1, \dots, u_{N-1} associated with a string $s_0s_1 \dots s_{N-1}$ that generates a periodic sequence admissible adjacent to zero is subadditive, in the sense that

$u_{m+n} \leq u_m + u_n$, provided $u_0 = 1$ (i.e. provided the string is cycled so that $s_0 = +$.)

Proof

Restating from (9.1) the inequalities satisfied by the u_r 's, we have

$$-\frac{1}{2}(N-1) \leq y - \left(\frac{1}{2}(N-1) - r\right)x - Nu_r \leq \frac{1}{2}(N-1) \quad \text{for } r = 0, 1, \dots, N-1.$$

The inequality for $r = m+n$ gives

$$Nu_{m+n} \leq y - \frac{1}{2}(N-1)x + mx + nx + \frac{1}{2}(N-1) \quad (9.2)$$

and correspondingly the inequalities for $r = m$ and for $r = n$ give

$$\begin{aligned} mx &\leq -y + \frac{1}{2}(N-1)x + Nu_m + \frac{1}{2}(N-1), \\ nx &\leq -y + \frac{1}{2}(N-1)x + Nu_n + \frac{1}{2}(N-1). \end{aligned} \quad (9.3)$$

Substituting (9.3) into (9.2) gives

$$Nu_{m+n} \leq Nu_m + Nu_n - y + \frac{1}{2}(N-1)x + \frac{3}{2}(N-1).$$

From the inequality for $r = 0$,

$$-y + \frac{1}{2}(N-1)x \leq \frac{1}{2}(N-1) - Nu_0$$

so

$$Nu_{m+n} \leq Nu_m + Nu_n - Nu_0 + 2(N-1).$$

If now $u_0 = 1$, then

$$u_{m+n} \leq u_m + u_n + \frac{N-2}{N}, \quad (9.4)$$

from which it follows that $u_{m+n} \leq u_m + u_n$, because all the u_r 's are integers. \square

Remark : In fact the subadditivity $u_{m+n} \leq u_m + u_n$ is true even when $m+n$ exceeds the length of the string, because the inequalities (9.1) continue to hold when $r \geq N$ if we define the u_r 's appropriately. Concatenating two copies of the string, $s_0 s_1 \dots s_{N-1} s_0 s_1 \dots s_{N-1}$, then the t_r 's repeat, because $t_{N-1} = 0$. If $r = N + i$ then $u_r = u_{N-1} + u_i = x + u_i$ so

$$y - \left(\frac{1}{2}(N-1) - r\right)x - Nu_r = y - \left(\frac{1}{2}(N-1) - i\right)x - Nu_i. \quad (9.5)$$

In the proof of the proposition it is conceivable that some of the indices could exceed the string length N and this remark allows for that possibility.

Proof of Proposition

We start by establishing a simpler result that involves two runs of zeros separated by a single $+-$ pair. Let m be the length of any run of zeros in the string $s_0 s_1 \dots s_{N-1}$ and n be the length of the next run of zeros beyond an intervening $+-$ pair. We show that $n > m - 2$.

By cycling we can assume that the string $s_0 s_1 \dots s_{N-1}$ starts with a $+-$ pair followed by the run of m zeros. Thus the start of the string can be written

$$+ - \underbrace{0 \dots 0}_{m \text{ zeros}} + - \underbrace{0 \dots 0}_{n \text{ zeros}} +$$

and the progression of u_r 's at the start of the string is

| r | 0 | 1 | 2 | ... | $m+1$ | $m+2$ | $m+3$ | $m+4$ | ... | $m+n+3$ | $m+n+4$ |
|-------|---|---|---|-----|-------|-------|-------|-------|-----|---------|---------|
| s_r | + | - | 0 | ... | 0 | + | - | 0 | ... | 0 | + |
| u_r | 1 | 1 | 1 | ... | 1 | 2 | 2 | 2 | ... | 2 | 3 |

By the subadditivity property of the u_r 's, $u_{2m+2} \leq u_{m+1} + u_{m+1} = 2$. Since $u_{m+n+4} = 3$ and the u_r 's form an increasing progression, it must be the case that $m+n+4 > 2m+2$, so $n > m - 2$. In particular note that this argument allows for the possibility $n = 0$, so that a run of two or more zeros is followed by at most a single $+-$ pair.

We will now use this result to prove that the run-lengths of zeros come only in two sizes, which differ by 1. Let m denote the length of the longest run of zeros in $s_0 s_1 \dots s_{N-1}$, and suppose that there were also a run of zeros of length $\leq m - 2$. By the result we have just

established, there must be a section of the string (possibly involving wrap-around) which consists of a run of m zeros, then one or more runs of $m - 1$ zeros followed by a run of $\leq m - 2$ zeros. By cycling we can suppose that this section of the string occurs at the start, so that the string $s_0 s_1 \dots s_{N-1}$ commences

$$+ - \underbrace{0 \dots 0}_{m \text{ zeros}} + - \underbrace{0 \dots 0}_{m-1 \text{ zeros}} + - \underbrace{0 \dots 0}_{m-1 \text{ zeros}} \dots + - \underbrace{0 \dots 0}_{m-1 \text{ zeros}} + - \underbrace{0 \dots 0}_{\leq m-2 \text{ zeros}} +$$

repeated k times

If n denotes the length of this final run of zeros, the corresponding progression of the u_r 's is

| r | 0 | 1 | 2 | ... | $m+1$ | $m+2$ | ... | ... | $(m+2)+k(m+1)+(n+2)$ |
|-------|---|---|---|-----|-------|-------|-----|-----|----------------------|
| s_r | + | - | 0 | ... | 0 | + | ... | ... | + |
| u_r | 1 | 1 | 1 | ... | 1 | 2 | ... | ... | $k+3$ |

Because of subadditivity, $u_{(k+2)(m+1)} \leq u_{m+1} + u_{m+1+k(m+1)} = 1 + (k+1) = k+2$. But $u_{(m+2)+k(m+1)+(n+2)} = k+3$, so $(k+2)(m+1) < (k+1)(m+1) + n+3$, i.e. $n > m-2$. Thus there can be no run of zeros of length $\leq m-2$ and the string $s_0 s_1 \dots s_{N-1}$ only has runs of zeros of length $m-1$ and m . \square

That all the strings which generate periodic sequences admissible adjacent to zero have runs of just two different lengths is evident when we look at the lists of strings, for example Table 7.2 on page 316 or the list for $\theta = 0.15$ of Table 1 in Volume II, but this structural feature did not arise naturally out of the classification of these strings described in Chapter 7. Note that the result holds equally well when one of the run lengths of zeros is itself zero. This means that if a run of at least two 0's appears in a string, then all the $+-$ pairs are isolated, and conversely if the substring $+-+-$ is present then all the 0's are isolated.

Using the knowledge that the runs of zeros in a string only appear in two run lengths, which differ by 1, we can calculate the run lengths of zeros for a general string of length N

with x $+-$ pairs. Write the division of N by x in quotient and remainder form

$$N = qx + r, \quad \text{where } q = \left\lfloor \frac{N}{x} \right\rfloor \text{ and } 0 \leq r < x.$$

Every $+-$ pair is followed by a run of either $q - 2$ or $q - 1$ zeros, and r of these runs have length $q - 1$.

Rather than seeking to identify further features of strings which generate periodic sequences admissible adjacent to zero, we describe how the ideas introduced here may serve as the basis for a second method of producing the strings. Note that the substrings of length q or $q + 1$ consisting of a $+-$ pair followed by a run of zeros can themselves be considered as the building blocks of the string, in place of the original $+-$ pairs and 0 's. Denote by the letter S a substring consisting of a $+-$ pair followed by $q - 2$ zeros, and by S' a $+-$ pair followed by $q - 1$ zeros. We can replace the task of determining the arrangement of x $+-$ pairs and $(N - 2x)$ 0 's by that of determining the arrangement of $(x - r)$ S 's and r S' 's.

To be definite, suppose more S 's than S' 's appear. Then the S' 's play the role of the $+-$ pairs previously, and we need to know how the (more numerous) S 's are distributed between them. Study of our lists of strings that generate periodic sequences admissible adjacent to zero (and also further theoretical investigation) indicates that the S 's are distributed according to the same rule that we obtained above for the runs of zeros. That is, the runs of S 's are of (at most) two different lengths which differ by 1. But then we may calculate the lengths of runs of S 's and amalgamate each into a yet larger building block consisting of an S' followed by a run of S 's. Repeated application of this procedure gives a means of constructing the string : at each stage the more numerous symbols are distributed between the less numerous, and there are two run lengths which differ by 1. We provide two examples; the first includes the step-by-step details by way of explanation, the second is a larger example which better illustrates the recursive structure present in the string.

Example 9.1 : $N = 37, x = 11$

The string to be constructed consists of 11 $+-$ pairs, and $37 - 2 \times 11 = 15$ zeros. Because

$$37 = 3 \times 11 + 4,$$

of the 11 substrings consisting of a $+-$ pair and a run of zeros, there are 7 substrings $S = +-0$ and 4 substrings $S' = +-00$. We now match runs of S 's to the individual S' 's :

$$11 = 2 \times 4 + 3,$$

so there are 3 substrings $T = S'SS$ and one substring $T' = S'S$. Since there is just a single T' , up to cycling the T' and T 's can only be arranged in one way, namely $T'TTT$. The string is obtained in terms of its $+-$ pairs and 0's by back substitution :

$$\begin{aligned} T'TTT &= S'S S'SS S'SS S'SS \\ &= +-00 +-0 +-00 +-0 +-0 +-00 +-0 +-0 +-00 +-0 +-0. \end{aligned}$$

Example 9.2 : $N = 101, x = 30$

$$\begin{aligned} 101 &= 3 \times 30 + 11 &\Rightarrow & 19 \text{ of } S = +-0 \text{ and } 11 \text{ of } S' = +-00 \\ 30 &= 2 \times 11 + 8 &\Rightarrow & 8 \text{ of } T = S'SS \text{ and } 3 \text{ of } T' = S'S \\ 11 &= 3 \times 3 + 2 &\Rightarrow & 2 \text{ of } U = T'TTT \text{ and } 1 \text{ of } U' = T'TT \end{aligned}$$

So the string is

$$\begin{aligned} U'UU &= T'TT T'TTT T'TTT \\ &= S'S S'SS S'SS S'S S'SS S'SS S'SS S'SS S'SS S'SS \\ &= +-00 +-0 +-00 +-0 +-0 +-00 +-0 +-0 +-00 +-0 +-00 +-0 \\ &\quad +-0 +-00 +-0 +-0 +-00 +-0 +-0 +-00 +-0 +-00 +-0 +-0 \\ &\quad +-00 +-0 +-0 +-00 +-0 +-0. \end{aligned}$$

The procedure necessarily terminates (indeed, the calculation is one variant of the Euclidean algorithm) and produces a unique string for each pair N and x .

9.2 The functions associated to the Chua-Lin inequalities

Recall that to a string $s_0 s_1 \dots s_{N-1}$ that generates a periodic sequence admissible adjacent to zero, we have associated the function

$$\begin{aligned} f_0(\theta) &= s_0 \cos \frac{N}{2} \theta + s_1 \cos \frac{N-2}{2} \theta + \dots + s_{N-1} \cos \frac{N-2}{2} \theta \\ &= \sum_{r=0}^{N-1} s_r \cos \frac{N-2r}{2} \theta \end{aligned} \quad (9.6)$$

and also functions f_1, f_2, \dots, f_{N-1} obtained in parallel fashion from the cycled versions of the string. These N trigonometric polynomials are the subjects constrained by the Chua-Lin inequalities (see Section 2.1.4).

Proposition 9.3

When x and N are coprime and the string $s_0 s_1 \dots s_{N-1}$ arises from the construction of Section 7.5, so that $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$ for $r = 0, 1, \dots, N-1$, the function f_0 is

$$f_0(\theta) = \cos \frac{N}{2} \theta - \cos \frac{N-2}{2} \theta. \quad (9.7)$$

Proof

It is sufficient to establish that if $s_r = +$ then $s_{N-1-r} = -$ for $r \neq 0$. This is because the contribution $\cos \frac{N-2r}{2} \theta - \cos \frac{N-2r-2}{2} \theta$ to f_0 from a $+-$ pair is cancelled by the contribution

$$\cos \frac{N-2(N-1-r)}{2} \theta - \cos \frac{N-2(N-1-r)-2}{2} \theta = \cos \frac{N-2r-2}{2} \theta - \cos \frac{N-2r}{2} \theta, \quad (9.8)$$

and so $f_0 = s_0 \cos \frac{N}{2} \theta + s_1 \cos \frac{N-2}{2} \theta + s_{N-1} \cos \frac{N-2}{2} \theta = \cos \frac{N}{2} \theta - \cos \frac{N-2}{2} \theta$, since it is easily checked that $s_0 = +$, $s_1 = -$ and $s_{N-1} = 0$.

Suppose then $s_r = +$. But this is when $u_r - u_{r-1} = 1$. The formula $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$ means that Nu_r is the unique multiple of N in the range

$$(r+1)x, (r+1)x+1, \dots, (r+1)x+(N-1),$$

and that Nu_{r-1} is the unique multiple of N in the range

$$rx, rx + 1, \dots, rx + (N - 1).$$

If $u_r = u_{r-1} + 1$, then Nu_{r-1} is in the range

$$rx, rx + 1, \dots, rx + (x - 1).$$

Now rx and $(r + 1)x$ are not multiples of N , otherwise x and N are not coprime. So there is a multiple of N in the range

$$rx + 1, \dots, (r + 1)x;$$

consequently there is a multiple of N in the range

$$\begin{array}{ccc} Nx - (r + 1)x, & \dots, & Nx - (rx + 1). \\ = (N - 1 - r)x & & = (N - r)x - 1 \end{array}$$

But Nu_{N-1-r} is the unique multiple of N in the range

$$(N - r)x, (N - r)x + 1, \dots, (N - r)x + (N - 1),$$

and Nu_{N-2-r} is the unique multiple of N in the range

$$(N - 1 - r)x, (N - 1 - r)x + 1, \dots, (N - 1 - r)x + (N - 1).$$

So Nu_{N-2-r} is in the range $(N - 1 - r)x, \dots, (N - r)x - 1$, and $u_{N-1-r} - u_{N-2-r} = 1$, therefore $s_{N-1-r} = +$. □

In the foregoing proof it was not necessary to suppose that N is odd. This is an essential assumption however for our next proposition, which is concerned with finding the zero function amongst f_0, f_1, \dots, f_{N-1} .

Proposition 9.4

If N is odd then some f_r is the zero function.

Proof

With $y = \frac{1}{2}(N+1)x$ we find that y lies between $y_{\min} = \frac{1}{2}(N+3)x - \frac{1}{2}(N-1)$ and $y_{\max} = \frac{1}{2}(N+1)x + \frac{1}{2}(N-1)$. If h is the highest common factor of x and N , and $x = ah$, $N = mh$ then

$$\frac{1}{2}(N+1)x + \frac{1}{2}(N-h) - y = \frac{1}{2}(N-h) = \frac{1}{2}(m-1)h, \quad (9.9)$$

i.e. a quantity divisible by h . It follows from Section 7.6 that we may cycle the string so that if the cycled version is written $s_0s_1 \dots s_{N-1}$ and the u_r 's are correspondingly determined, then

$$-\frac{1}{2}(N-1) \leq y - \left(\frac{1}{2}(N-1) - r\right) - Nu_r \leq \frac{1}{2}(N-1) \quad (9.10)$$

in which $y = \frac{1}{2}(N+1)x$, and so

$$(r+1)x - \frac{1}{2}(N-1) \leq Nu_r \leq (r+1)x + \frac{1}{2}(N-1). \quad (9.11)$$

We show that a consequence of these inequalities is that f_0 is the zero function.

Note first that $s_0 = 0$ follows from the inequality for $r = 0$, because $x \leq \frac{1}{2}(N-1)$. We show below, in a manner similar to the proof of Proposition 9.3, that if $s_r = +$ then $s_{N-1-r} = +$ for $r \neq 0$. But then $s_r = +$ if and only if $s_{N-r-1} = +$, if and only if $s_{N-r} = -$, and $s_r = -$ if and only if $s_{r-1} = +$, if and only if $s_{N-r+1} = -$, if and only if $s_{N-r} = +$. Thus $s_r + s_{N-r} = 0$ for $r \neq 0$, and

$$f_0(\theta) = \sum_{r=0}^{N-1} s_r \cos \frac{N-2r}{2} \theta = s_0 \cos \frac{N}{2} \theta + \sum_{r=1}^{\frac{1}{2}(N-1)} (s_r + s_{N-r}) \cos \frac{N-2r}{2} \theta = 0. \quad (9.12)$$

Suppose then $s_r = +$, so that $u_r - u_{r-1} = 1$. But Nu_r is the unique multiple of N in the range $(r+1)x - \frac{1}{2}(N-1)$ to $(r+1)x + \frac{1}{2}(N-1)$, and Nu_{r-1} is the unique multiple of N in

the range $rx - \frac{1}{2}(N-1)$ to $rx + \frac{1}{2}(N-1)$. Since u_r and u_{r-1} differ, Nu_{r-1} is in the range $rx - \frac{1}{2}(N-1)$ to $rx - \frac{1}{2}(N-1) + (x-1)$. So there is a multiple of N in the range from

$$N(x-1) - rx + \frac{1}{2}(N-1) - (x-1) = (N-1-r)x - \frac{1}{2}(N-1)$$

to

$$N(x-1) - rx + \frac{1}{2}(N-1) = (N-r)x - \frac{1}{2}(N-1) - 1.$$

This multiple of N is Nu_{N-r-2} , but not Nu_{N-r-1} , and thus $u_{N-r-1} - u_{N-r-2} = 1$, that is $s_{N-r-1} = +$. \square

Thus we have seen that a string that generates a periodic sequence admissible adjacent to zero having length N and $x+-$ pairs, where N is odd and x is coprime to N , has amongst its associated functions f_r the function $\cos \frac{N}{2}\theta - \cos \frac{N-2}{2}\theta$ and the zero function. To turn this about, we can ask which odd length strings include these two functions amongst the associated trigonometric function set.

To illustrate the point raised here let us consider strings of length 11. Since some f_r is to be the function $\cos \frac{11}{2}\theta - \cos \frac{9}{2}\theta$, we may cycle the string so that it is f_0 . Then

$$\begin{aligned} f_0(\theta) &= s_0 \cos \frac{11}{2}\theta + (s_1 + s_{10}) \cos \frac{9}{2}\theta + (s_2 + s_9) \cos \frac{7}{2}\theta + (s_3 + s_8) \cos \frac{5}{2}\theta + \\ &\quad (s_4 + s_7) \cos \frac{3}{2}\theta + (s_5 + s_6) \cos \frac{1}{2}\theta \\ &= \cos \frac{11}{2}\theta - \cos \frac{9}{2}\theta, \end{aligned} \quad (9.13)$$

so that $s_0 = +$, one of s_1 and s_{10} is $-$, the other 0, and $s_r = -s_{N-r}$ for $r = 2, 3, 4, 5$. After appropriate cycling and negation, the string is thus

$$+ \quad - \quad s_2 \quad s_3 \quad s_4 \quad s_5 \quad -s_5 \quad -s_4 \quad -s_3 \quad -s_2 \quad 0.$$

But a further cycling is to be possible so that the corresponding f_0 is the zero function : this is so for $s'_0 s'_1 \dots s'_{N-1}$ when $s'_0 = 0$ and $s'_r = -s'_{N-r}$ for $r = 1, 2, 3, 4, 5$. Either the 0 at

the end is cycled to the start, or one of $\pm s_2, \pm s_3, \pm s_4$ or $\pm s_5$ is 0 and cycled to the start. Consider the possibilities :

1. 0 at the end cycled to the start :

| | | | | | | | | | | |
|---|---|---|-------|-------|-------|-------|--------|--------|--------|--------|
| | A | B | C | D | E | E | D | C | B | A |
| 0 | + | - | s_2 | s_3 | s_4 | s_5 | $-s_5$ | $-s_4$ | $-s_3$ | $-s_2$ |

and pairing the coefficients of like cosine terms via the labels, we find $s_2 = +$, $s_3 = -$, $s_4 = +$, $s_5 = -$, so the original string is $+-+--+-+0$.

2. $s_2 = -s_2 = 0$ cycled to the start : this case produces the string $+0+-00+-00$.
3. $s_3 = -s_3 = 0$ cycled to the start : this case produces the string $+000+-0000$.
4. $s_4 = -s_4 = 0$ cycled to the start : this case produces the string $+-+0+-0+-0$.
5. $s_5 = -s_5 = 0$ cycled to the start : this case produces the string $+000000000$.

That is we obtain precisely the length 11 strings that generate periodic sequences admissible adjacent to zero.

9.3 Intervals of admissibility

We have seen in the previous section that any *irreducible* string of length N which generates a periodic sequence admissible adjacent to zero has the function $\cos \frac{N}{2}\theta - \cos \frac{N-2}{2}\theta$ amongst its associated set of N trigonometric polynomials. (For the moment, purely to avoid complicating unnecessarily our discussion and the statement of the proposition that follows, we assume that N is odd : all that we show persists in the even case, indeed the discussion and proof are identical, except that wherever we mention "a periodic sequence admissible adjacent to zero", the latter must be replaced by "a periodic sequence generated by a string arising by the rule $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$ ", as we did in Proposition 9.3, previously. Of course the source of this awkwardness is the fact that when N is even we have not been able to establish in full generality the equivalence of the two notions (cf. Chapter 8).) This

function occurred previously, in Chapter 5, when we considered parameter values θ where the length N string $+-0\dots 0$ generates a periodic sequence admissible adjacent to zero. In particular we showed that $\cos \frac{N}{2}\theta - \cos \frac{N-2}{2}\theta$ stays within the envelope of $\pm g(\theta)$ for $0 < \theta < \frac{\pi}{N}$, and that this interval is maximal. Since any periodic sequence can be generated by an irreducible string, it follows that for any admissible periodic sequence of least period N whose interval of admissibility has left end-point zero, the right end-point of the interval cannot fall to the right of $\frac{\pi}{N}$. We know already for the sequence generated by the length N string $+-0\dots 0$ that the interval of admissibility is precisely $(0, \frac{\pi}{N})$ (see Propositions 5.5 and 5.6). Additionally all the periodic sequences admissible adjacent to zero found by our computer searches have empirically determined intervals of admissibility agreeing with the prescription $(0, \frac{\pi}{N})$. Our next proposition establishes quite generally that $(0, \frac{\pi}{N})$ is the interval of admissibility for each periodic sequence admissible adjacent to zero. It is preceded by an elementary lemma about the cosine function, which is needed for the proof of the proposition. Before that though we present an example that enables us to indicate in outline the argument we use to prove the proposition.

Example 9.3 : The functions f_0, f_1, \dots, f_{10} for the string $+-0+-00+-00$

The string considered here arises for the choice $N = 11$ and $x = 3$. The 11 trigonometric polynomial functions f_0, f_1, \dots, f_{10} correspond to the cycled versions of the string as follows :

| | | |
|----------------|-------------------|----------|
| $+-0+-00+-00$ | \longrightarrow | f_0 |
| $-0+-00+-00+$ | \longrightarrow | f_1 |
| $0+-00+-00+-$ | \longrightarrow | f_2 |
| $+ -00+-00+-0$ | \longrightarrow | f_3 |
| $-00+-00+-0+$ | \longrightarrow | f_4 |
| $00+-00+-0+-$ | \longrightarrow | f_5 |
| $0+-00+-0+-0$ | \longrightarrow | f_6 |
| $+ -00+-0+-00$ | \longrightarrow | f_7 |
| $-00+-0+-00+$ | \longrightarrow | f_8 |
| $00+-0+-00+-$ | \longrightarrow | f_9 |
| $0+-0+-00+-0$ | \longrightarrow | f_{10} |

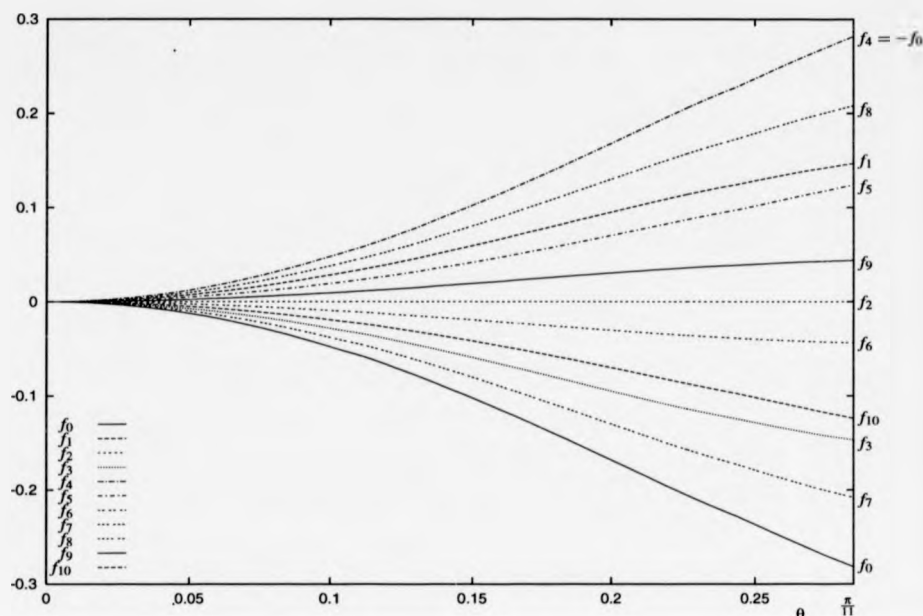
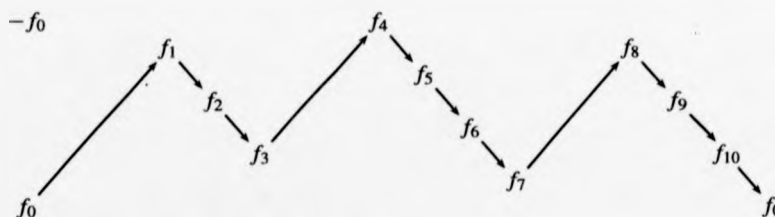


Figure 9.1: Plots of f_0, f_1, \dots, f_{10} and $-f_0$ for the string $+-0+-00+-0$ to exhibit the order relations that prevail over the interval $(0, \frac{\pi}{11})$.

Plots of f_0, f_1, \dots, f_{10} and $-f_0$ over the range $(0, \frac{\pi}{11})$ are shown in Figure 9.1. Observe that f_3, f_7 correspond to a cycled state of the string commencing with a $+-$ pair and that each exceeds f_0 , that f_1, f_4, f_8 correspond to a cycled state of the string commencing with a $-$ symbol and that each is exceeded by $-f_0$, and that $f_1 \geq f_2 \geq f_3, f_4 \geq f_5 \geq f_6 \geq f_7$, and $f_8 \geq f_9 \geq f_{10} \geq f_0$. These order relations between f_0, f_1, \dots, f_{10} can be represented in the diagrammatic form :



The proof of our proposition proceeds by establishing in the general case the order relations between the f_r 's corresponding to those remarked upon here.

Lemma 9.5

If $-\frac{\pi}{2} \leq \alpha_1 \leq \beta_1 \leq \frac{\pi}{2}$, $-\frac{\pi}{2} \leq \alpha_2 \leq \beta_2 \leq \frac{\pi}{2}$, $\beta_1 - \alpha_1 = \beta_2 - \alpha_2$ and $\alpha_1 < \alpha_2$, then

$$\cos \alpha_2 - \cos \beta_2 > \cos \alpha_1 - \cos \beta_1. \quad (9.14)$$

Proof

$$\cos \alpha_1 - \cos \beta_1 = 2 \sin \frac{\alpha_1 + \beta_1}{2} \sin \frac{\beta_1 - \alpha_1}{2} < 2 \sin \frac{\alpha_2 + \beta_2}{2} \sin \frac{\beta_2 - \alpha_2}{2} = \cos \alpha_2 - \cos \beta_2. \quad \square$$

Proposition 9.6

Each periodic sequence admissible adjacent to zero with least period N has interval of admissibility $(0, \frac{\pi}{N})$.

Proof

Suppose $s_0 s_1 \dots s_{N-1}$ is a length N generating string, that has x $+-$ pairs where x and N are coprime because of irreducibility. In Proposition 9.3 we showed that if the string is assumed cycled to the standard configuration (i.e. that implicit in the construction of Section 7.5) then $f_0(\theta) = \cos \frac{N}{2}\theta - \cos \frac{N-2}{2}\theta$.

(a) Consider f_r arising from the string $s_0 s_1 \dots s_{N-1}$ cycled to $s_r s_{r+1} \dots s_{N-1} s_0 s_1 \dots s_{r-1}$ and such that $s_r = +$, $s_{r+1} = -$. We show that $f_0 \leq f_r$ over $(0, \frac{\pi}{N})$.

In what follows u_0, u_1, \dots, u_{N-1} relate to the string in this cycled state. With $y = u_0 + u_1 + \dots + u_{N-1}$, set $\delta = y_{\max} - y$. Then $1 \leq \delta \leq x - 1$ because $u_0 = 1$ and $u_i = \left\lceil \frac{(i+1)x - \delta}{N} \right\rceil$. Suppose $u_i - u_{i-1} = 1$. Now Nu_{i-1} is in the integer range $ix - \delta$ to $ix - \delta + (N - 1)$, Nu_i is in the integer range $(i+1)x - \delta$ to $(i+1)x - \delta + (N - 1)$, so Nu_{i-1} is within $ix - \delta$ to $ix - \delta + (x - 1)$. It follows that the integer range

$$\text{range } 0: (N-i)x + \delta - (x-1) \quad \text{to} \quad (N-i)x + \delta$$

contains a multiple of N .

First case : $1 \leq \delta \leq \frac{1}{2}(x-1)$

Consider the three overlapping integer ranges :

range 1 : $(N-i)x - x - \delta$ to $(N-i)x - x - \delta + (N-1)$ which includes Nu_{N-2-i} ,

range 2 : $(N-i)x - \delta$ to $(N-i)x - \delta + (N-1)$ which includes Nu_{N-1-i} ,

range 3 : $(N-i)x + x - \delta$ to $(N-i)x + x - \delta + (N-1)$ which includes Nu_{N-i} .

The total span from $(N-i)x - x - \delta$ to $(N-i)x + x - \delta + (N-1)$ is

$$(N-i)x + x - \delta + (N-1) - ((N-i)x - x - \delta - 1) = 2x + 1 + (N-1) \leq 2N - 1$$

so contains exactly two multiples of N (using $2x \leq N-1$, which follows because x and N are coprime). Now for δ restricted as specified, range 0 is contained within range 1 and does not meet range 3. It follows that *either* $u_{N-2-i} = u_{N-1-i}$ and $u_{N-i} - u_{N-1-i} = 1$, *or* $u_{N-1-i} - u_{N-2-i} = 1$ and $u_{N-1-i} = u_{N-i}$.

The function f_r arises from the string $s_0 s_1 \dots s_{N-1}$ cycled to $s_r s_{r+1} \dots s_{N-1} s_0 s_1 \dots s_{r-1}$. Suppose in this cycled state the string is

$$\begin{array}{cccccccccccccccccccc} + & - & 0 & \dots & 0 & + & - & 0 & \dots & 0 & + & - & 0 & \dots & 0 & \dots & + & - & 0 & \dots & 0 \\ & & & & & \uparrow & & & & & \uparrow & & & & & & & \uparrow & & & & \\ & & & & & i_1 & & & & & i_2 & & & & & & & i_{x-1} & & & & \end{array}$$

Then

$$\begin{aligned} f_r(\theta) = & \left(\cos \frac{N}{2} \theta - \cos \frac{N-2}{2} \theta \right) + \left(\cos \frac{N-2i_1}{2} \theta - \cos \frac{N-2i_1-2}{2} \theta \right) + \dots \\ & + \left(\cos \frac{N-2i_{x-1}}{2} \theta - \cos \frac{N-2i_{x-1}-2}{2} \theta \right). \end{aligned} \quad (9.15)$$

Consider an index i from i_1, i_2, \dots, i_{x-1} , but with $2i < N-1$. It affords the contribution

$$\cos \frac{N-2i}{2} \theta - \cos \frac{N-2i-2}{2} \theta \quad (9.16)$$

to f_r . But then the term

$$\cos \frac{N-2(N-i)}{2} \theta - \cos \frac{N-2(N-i)-2}{2} \theta = \cos \frac{N-2i}{2} \theta - \cos \frac{N-2i+2}{2} \theta \quad (9.17)$$

or the term

$$\cos \frac{N-2(N-1-i)}{2}\theta - \cos \frac{N-2(N-1-i)-2}{2}\theta = \cos \frac{N-2i-2}{2}\theta - \cos \frac{N-2i}{2}\theta \quad (9.18)$$

is present. Either of the latter terms contributes positively to the sum of f_r and because of Lemma 9.5 sufficiently to compensate for the negative $\cos \frac{N-2i}{2}\theta - \cos \frac{N-2i-2}{2}\theta$ contribution. Notice that indices i_1 and i_2 will be such that $i_1 + 1 < i_2$, so that $N - i_2 < N - 1 - i_1$, and each negative contribution to f_r is balanced by its own positive contribution. It follows then that $f_r \geq f_0$.

Second case : $\frac{1}{2}x \leq \delta \leq x - 1$

Consider instead the three overlapping integer ranges :

- range 1 : $(N-i)x - \delta$ to $(N-i)x - \delta + (N-1)$ which includes Nu_{N-1-i} ,
 range 2 : $(N-i)x + x - \delta$ to $(N-i)x + x - \delta + (N-1)$ which includes Nu_{N-i} ,
 range 3 : $(N-i)x + 2x - \delta$ to $(N-i)x + 2x - \delta + (N-1)$ which includes Nu_{N+1-i} .

As before the total span contains exactly two multiples of N , the range 0 is contained within range 1, but does not meet range 3. So *either* $u_{N-1-i} = u_{N-i}$ and $u_{N+1-i} - u_{N-i} = 1$, *or* $u_{N-i} - u_{N-1-i} = 1$ and $u_{N-i} = u_{N+1-i}$. It follows again that $f_r \geq f_0$.

(b) As in (a) consider the string $s_0s_1 \dots s_{N-1}$ cycled to $s_rs_{r+1} \dots s_{N-1}s_0s_1 \dots s_{r-1}$ and such that $s_r = +$, $s_{r+1} = -$, but we show that $f_{r+1} \leq -f_0$ over $(0, \frac{\pi}{N})$.

Consider an index i from i_1, i_2, \dots, i_{x-1} , but with $2i > N - 1$. The corresponding $+-$ pair contributes

$$\cos \frac{2i-N-2}{2}\theta - \cos \frac{2i-N}{2}\theta \quad (9.19)$$

to f_{r+1} . From the arguments in part (a) :

First case : $1 \leq \delta \leq \frac{1}{2}(x-1)$

Either $s_{N-i} = +$ or $s_{N-1-i} = +$, so either

$$\cos \frac{N-2(N-i)+2}{2}\theta - \cos \frac{N-2(N-i)}{2}\theta = \cos \frac{2i-N+2}{2}\theta - \cos \frac{2i-N}{2}\theta \quad (9.20)$$

or

$$\cos \frac{N-2(N-1-i)+2}{2}\theta - \cos \frac{N-2(N-1-i)}{2}\theta = \cos \frac{2i-N+4}{2}\theta - \cos \frac{2i-N+2}{2}\theta \quad (9.21)$$

is present in f_{r+1} . They both contribute negatively and either will compensate for the positive $\cos \frac{2i-N-2}{2}\theta - \cos \frac{2i-N}{2}\theta$.

Second case : $\frac{1}{2}x \leq \delta \leq x-1$

Either $s_{N-i} = +$ or $s_{N+1-i} = +$, and so present in f_{r+1} is either $\cos \frac{2i-N+2}{2}\theta - \cos \frac{2i-N}{2}\theta$ or $\cos \frac{2i-N}{2}\theta - \cos \frac{2i-N-2}{2}\theta$. Again each will compensate for $\cos \frac{2i-N-2}{2}\theta - \cos \frac{2i-N}{2}\theta$.

It follows that $f_{r+1} \leq -f_0$, because the $+-$ pair split between the start and end of the string giving rise to f_{r+1} contributes $-\cos \frac{N}{2}\theta + \cos \frac{N-1}{2}\theta = -f_0$.

(c) Consider once more the function f_r arising from the string $s_0s_1 \dots s_{N-1}$ cycled to $s_rs_{r+1} \dots s_{N-1}s_0s_1 \dots s_{r-1}$, but where now $s_r = 0$ or $s_r = -$. We compare f_r and f_{r+1} and show that $f_{r+1} \leq f_r$ over $(0, \frac{\pi}{N})$.

The contribution of a $+-$ pair to f_r is

$$\cos \frac{N-2i}{2}\theta - \cos \frac{N-2i-2}{2}\theta, \quad (9.22)$$

and the contribution of that pair to $f_r - f_{r+1}$ is

$$\left(\cos \frac{N-2i}{2}\theta - \cos \frac{N-2i-2}{2}\theta\right) - \left(\cos \frac{N-2i+2}{2}\theta - \cos \frac{N-2i}{2}\theta\right). \quad (9.23)$$

This also works if $s_r = -$, because the $+-$ pair of which $s_r = -$ is the second component contributes to f_r

$$-\cos \frac{N}{2}\theta + \cos \frac{N-2}{2}\theta = \cos \frac{N-2(N-1)}{2}\theta - \cos \frac{N-2(N-1)-2}{2}\theta \quad (9.24)$$

(as though the $-$ digit came at the end of the string). Now, each term contributing to $f_r - f_{r+1}$ is, according to the lemma, positive. Consequently $f_{r+1} \leq f_r$. \square

9.4 An application to the orbits in state-space : a measure theoretic bound

Our understanding of the periodic sequences admissible adjacent to zero permits an estimate of the size of the region in state-space made up of the points that generate periodic symbolic sequences. We saw in Section 2.1.3 that, for a given parameter value θ , the points in the state-space attributable to a given admissible periodic sequence of least period N fill out N solid ellipses, that these ellipses have equal areas because they are translates of one another, and that the area of each is

$$\pi \rho^2 \sin \theta \quad (9.25)$$

where $\rho = \frac{1}{|g(\theta)|} \min_{0 \leq r \leq N-1} (|g(\theta)| - f_r(\theta), |g(\theta)| + f_r(\theta))$. We should be slightly more specific on two counts here. Whereas throughout most of the thesis it has been convenient to view an admissible periodic sequence as defined by its generating string, and consequently we have regarded the N shifted versions of the sequence as essentially the same, here we need to recognise that these N shifted versions are associated individually with the N ellipses. Our second remark relates to the quantity ρ : ρ was defined immediately prior to Proposition 2.2, but the definition there is not our present formula, however the essence of the derivation of the Chua-Lin inequalities in Section 2.1.4 is to establish the equivalence of the two.

Consider a parameter choice close to 0. Whenever N is such that $\theta < \frac{\pi}{N}$, there are at least $\phi(N)$ periodic sequences of least period N that are admissible for the parameter value θ , those we have shown to have interval of admissibility $(0, \frac{\pi}{N})$. For each such sequence we saw in the course of the proof of Proposition 9.6 that $f_0(\theta) \leq f_r(\theta) \leq -f_0(\theta)$ for all r . If we recall that $g(\theta) = \sin \frac{N}{2}\theta \sin \theta$ and $f_0(\theta) = \cos \frac{N}{2}\theta - \cos \frac{N-2}{2}\theta$ then

$$\begin{aligned} \rho &= \frac{1}{\sin \frac{N}{2}\theta \sin \theta} (\sin \frac{N}{2}\theta \sin \theta + \cos \frac{N}{2}\theta - \cos \frac{N-2}{2}\theta) \\ &= \frac{1 - \cos \theta}{\sin \theta} \cot \frac{N}{2}\theta, \end{aligned} \quad (9.26)$$

in particular all the ellipses for all of these sequences have identical size. Figure 9.2 illustrating the case when $\theta = 0.3$ and $N = 5$, shows the 20 equal ellipses, in four sets of five, corresponding to the four strings $+-000$ (red), $-+000$ (green), $++--0$ (blue) and $-+-+0$ (yellow) which generate the period-5 sequences admissible adjacent to zero.

When θ is specified, the measure of the union of all these disjoint ellipses for all relevant N is

$$\mathcal{A}(\theta) = \pi \frac{(1 - \cos \theta)^2}{\sin \theta} \sum_{N < \frac{\pi}{\theta}} N \phi(N) \cot^2 \frac{N}{2} \theta. \quad (9.27)$$

This is clearly a lower bound for the measure of the set of points in state space associated with periodic symbolic sequences. This latter quantity has been the subject of investigation and conjecture in the literature : see in particular Ashwin (1996), Ashwin *et al.* (1997), Kocarev *et al.* (1996). We anticipate $\mathcal{A}(\theta)$ to be a good approximation when θ is close to zero, since our computer searches suggest that for these parameter values, the set of all admissible periodic sequences is largely accounted for by those that we have described as “admissible adjacent to zero”.

Figures 9.3 and 9.4, which relate to the parameter value $\theta = 0.3$, illustrate the level of agreement to expect. In Figure 9.3, the union of the blank ellipses corresponds to all the points associated with periodic symbolic sequences (we have exhibited these ellipses by plotting the complementary set of their union, as described in Ashwin (1996)). By comparison, Figure 9.4 shows the solid ellipses made up of the points generating 32 ($= \sum_{N < \frac{\pi}{\theta}} \phi(N)$) sequences admissible adjacent to zero when $\theta = 0.3$. To the eye, the two sets appear identical.

Some plots of $\mathcal{A}(\theta)$ for appropriate parameter ranges are given in Figures 9.5 and 9.6. Values of $\mathcal{A}(\theta)$ are to be judged against the total area of the state space which is 4.

ORIGINAL IN COLOUR

Figure 9.2 + 9.4



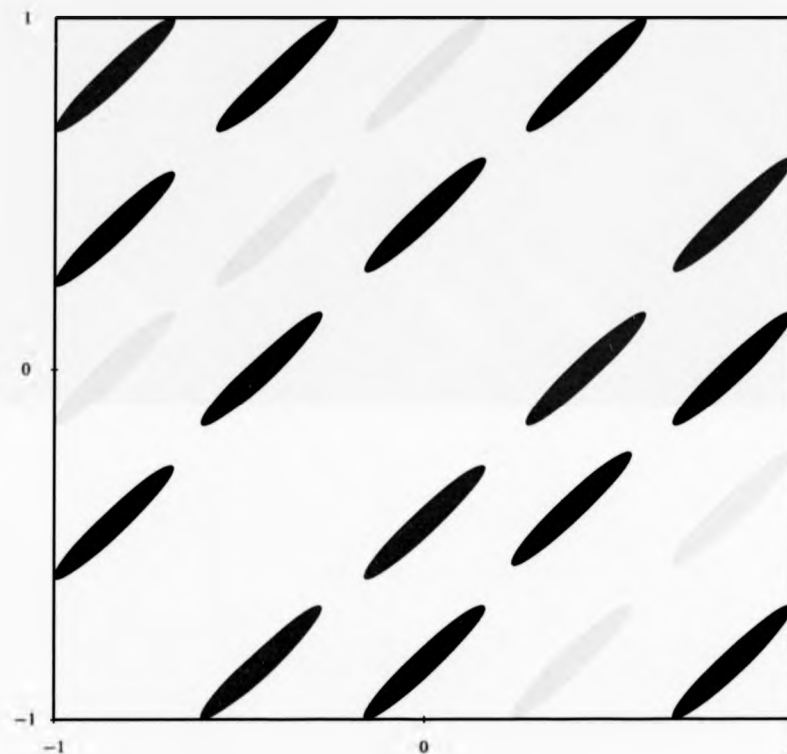


Figure 9.2: The points in I^2 that give rise to the period-5 sequences admissible adjacent to zero. (The parameter value for the plot is $\theta = 0.3 < \frac{\pi}{3}$.) The colours distinguish the four generating strings : $+-000$ (red), $-+000$ (green), $+-+-0$ (blue) and $-++0$ (yellow).

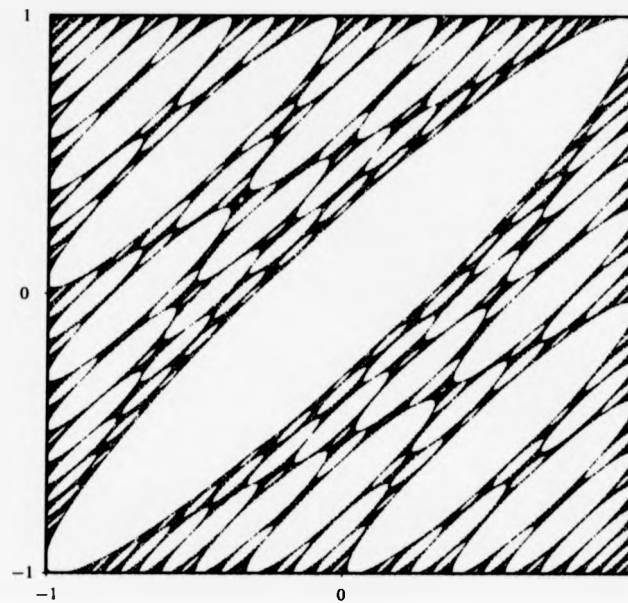


Figure 9.3: All the points giving rise to periodic sequences fill the unshaded region.

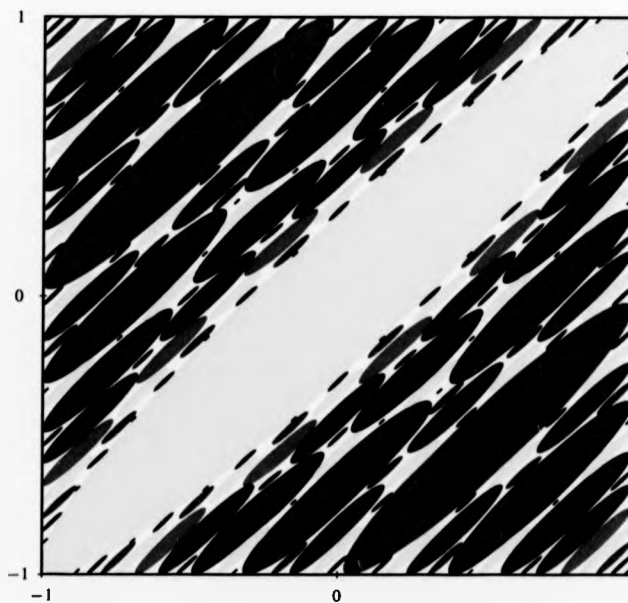


Figure 9.4: The solid ellipses used to form the bound $\mathcal{A}(\theta)$: all the points within them give rise to periodic sequences admissible adjacent to zero. Ellipses associated with sequences having the same period appear with the same colour.

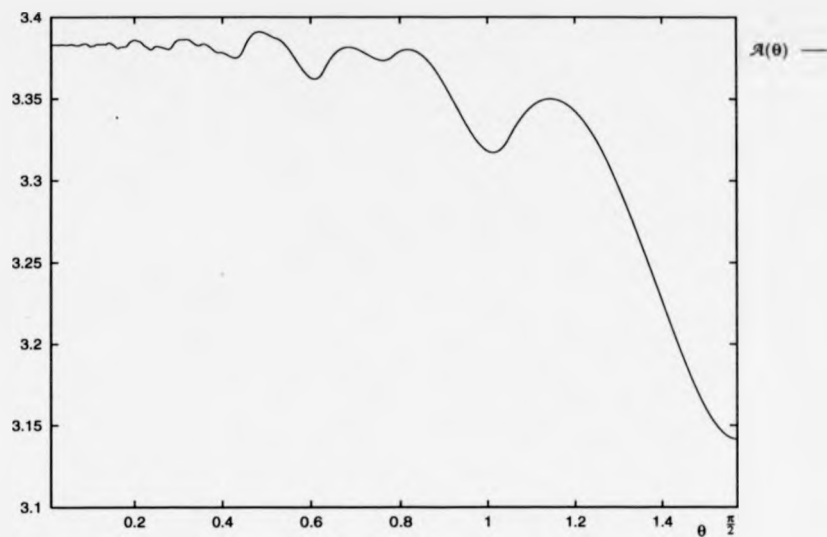


Figure 9.5: The graph of $\mathcal{A}(\theta)$, for $0 < \theta < \frac{\pi}{2}$; a lower bound for the measure of the subset of state-space consisting of the points that give rise to periodic sequences.

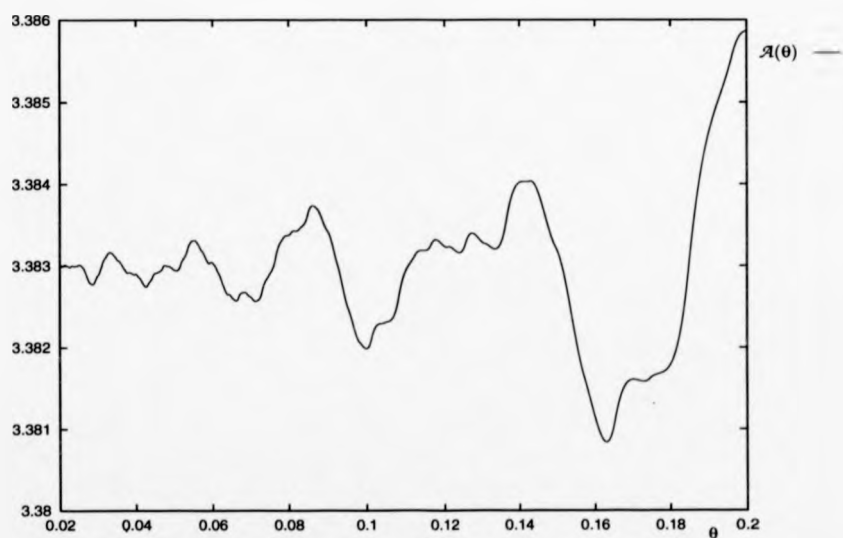


Figure 9.6: A magnification of the plot above, showing $\mathcal{A}(\theta)$ for parameter values close to zero where it may be expected to provide a good estimate.

Chapter Summary

- Within any string of odd length that generates a periodic sequence admissible adjacent to zero, a run of zeros separating $+-$ pairs (including the empty run between adjacent $+-$ pairs) can be of at most two run lengths, that differ by 1. The two types of sub-string formed by associating each $+-$ pair with its following run of zeros are distributed according to the same rule. Analysing the string structure via the hierarchy of the successive stages of combining compound string components gives an alternative collective description of the periodic sequences admissible adjacent to zero.
- Included amongst the suite of functions f_0, f_1, \dots, f_{N-1} , governed by the Chua-Lin inequalities, and associated with the string for which $u_r = \left\lceil \frac{(r+1)x}{N} \right\rceil$, are some specific evaluations :
 - When x and N are coprime, $f_0(\theta) = \cos \frac{N}{2}\theta - \cos \frac{N-2}{2}\theta$.
 - When N is odd some f_r is the zero function.

The requirement that both these should be present amongst the function set associated with a string $s_0 s_1 \dots s_{N-1}$ is shown, by example, to be a further way to characterise those odd length strings that generate periodic sequences admissible adjacent to zero.

- Each periodic sequence admissible adjacent to zero and with least period N has interval of admissibility $(0, \frac{\pi}{N})$ and this interval is maximal, whenever N is odd and for even N if we exclude anomalous sequences of the type examined in Chapter 8 (should any of these exist).
- The point sets in state-space that give rise to the periodic sequences of least period N corresponding to the characterisation in Chapters 7 and 8, are disjoint congruent ellipses. The combined area of the ellipses present at a given θ value,

$$\pi \frac{(1 - \cos \theta)^2}{\sin \theta} \sum_{N < \frac{\pi}{\theta}} N \phi(N) \cot^2 \frac{N}{2} \theta.$$

provides a lower bound for the total area of state-space occupied by points giving rise to periodic sequences.

Postscript

We conclude by reviewing some further lines of research suggested by the results of the investigations reported in the thesis.

There are evidently more families of admissible periodic sequences amenable to the type of analysis carried out in Chapters 5 and 6. The families investigated in Chapter 5 all had intervals of admissibility with an end-point at 0 , $\frac{\pi}{3}$, $\frac{\pi}{2}$ or π , and we have noticed that around many other rational multiples of π our computer searches for admissible periodic sequences have produced examples of sequences whose intervals of admissibility appear to extend as far as the rational multiple of π . We include one instance in Table 4 of Volume II; there the search is conducted at parameter values just to either side of $\frac{3\pi}{5}$. Both lists include sequences with interval of admissibility extending to $\frac{3\pi}{5}$, and notably a common pattern can be identified for the sequences having $\frac{3\pi}{5}$ as the left end-point of interval of admissibility. There are several such families distributed across the parameter range which seem worth investigating. Frequently, though, the step up in generating string length between successive sequences (as identified by the computer searches) is too great to propose with confidence a pattern for a family.

Although these families are numerous, and afford a good starting point for theoretical investigations, the majority of the admissible periodic sequences found by our computer searches, if we set to one side the abundant families admissible adjacent to zero or π , do not have interval end-points which are simple rational multiples of π . In this respect the sequences generated by the strings $+00+0\dots 0$, that we studied in Chapter 6, may be regarded as more representative of the "typical" admissible periodic sequence. Clearly one could easily take other simple string patterns and enquire whether they generate admissible

periodic sequences. We have applied our computer techniques from Chapter 6 to look at the following schemes for generating strings :

$$\begin{array}{ll}
 +0\dots 0+0\dots 0 & \text{i.e. } +0^k+0^l, \quad (k \neq l; \text{ the case } k=l \text{ is one of the} \\
 & \text{families at } \frac{\pi}{2} \text{ known from Chapter 5)} \\
 +- \dots +-0\dots 0 & \text{i.e. } (+-)^k 0^l \\
 +0\dots 0+0\dots 0+0\dots 0 & \text{i.e. } +0^k+0^l+0^m
 \end{array}$$

All were found to generate admissible periodic sequences for a large number of different string lengths, and indeed all share a distribution of intervals of admissibility qualitatively similar to that found in Chapter 6 for the generating strings $+00+0\dots 0$, though at markedly different locations within the parameter range $(0, \pi)$. As in Chapter 6, there is a complex array of intervals by means of which sub-families of sequences can be identified having interval locations that appear asymptotically to approach a rational multiple of π .

To conclude our discussion of families of sequences, we remark on two intriguing observations that emerged from a careful inspection of our computer lists of admissible periodic sequences. The first concerns admissible periodic sequences with interval of admissibility having an end-point that is coincident with a rational multiple of π , perhaps $\frac{p}{q}\pi$; in every instance we encountered, q divides the length of the generating string. Our second observation is of a possible symmetry amongst the admissible periodic sequences around $\frac{\pi}{3}$ (and likewise $\frac{2\pi}{3}$). There are many instances of these sequences with an interval of admissibility to the left of $\frac{\pi}{3}$, and not necessarily having $\frac{\pi}{3}$ as an end-point, for which the transformation that replaces components of the generating string like $(+-)^k 0^l$ by $(+-)^{k-1} + 0^{l+1}$, produces a new sequence admissible in some interval to the right of $\frac{\pi}{3}$, though in most cases the two intervals are not equidistant from $\frac{\pi}{3}$. An example of two strings conjugate in this way (with string length $N = 30$) is :

$$\begin{array}{ll}
 +--+--+0000--+0+-0+--+0000 & (0.878540634, 0.986253974) \\
 \downarrow & \\
 +--+--+00000--+00+00+--+00000 & (1.09476093, 1.17730656)
 \end{array}$$

We do not know to what extent either of these observations might form evidence for more general results.

Perhaps the main unresolved issue arising from the work in the thesis concerns establishing in complete generality for the case of the even length generating strings the classification of periodic sequences admissible adjacent to zero. It is interesting to note that the even length strings again present an obstruction in the work on periodic sequences admissible adjacent to $\frac{\pi}{2}$ reported in Appendix A. If the classification at zero is complete then we automatically obtain the classification at π , via the relation between the sequences admissible adjacent to zero and at π through the negation of alternate digits. A feature of the situation existing at π would be that all the sequences admissible adjacent to π necessarily have even length generating strings, apart from the three period-1 sequences involving a single repeated digit $+$, 0 and $-$. To illustrate this point, the irreducible string $s_0s_1s_2s_3s_4$ of length 5 would not be admissible adjacent to π for the following reason. For otherwise the string $s_0\bar{s}_1s_2\bar{s}_3s_4\bar{s}_0s_1\bar{s}_2s_3\bar{s}_4$ is admissible adjacent to zero, and in turn this contains only runs of 0 's and runs of $+-$ pairs. But then a first $+-$ pair is matched by a $-+$ pair in the second half of the length 10 string which must not be there for the string to be admissible adjacent to zero.

We mention an observation that connects the results of Chapter 9 with the study of the filter map in Region 3, as reported in Chapter 2. In the limit $\theta \rightarrow 0$, i.e. $a \rightarrow 2$, we obtain the relationship $b = -a + 1$, because for the map F we have been studying $b = -1$. So one could envisage a link to the dynamical behaviour that prevails for the map $F_{a,b}$ in the Region 3 as described in Section 2.3.1. There an orbit lies on a diagonal line wrapped back into the state space, see Figure 2.7(c), and for the map F when θ is close to zero we see in Figure 9.2 that an orbit jumps between a series of thin ellipses similarly aligned along a diagonal line wrapped back into the state-space.

Our final remarks concern the investigation of the one-dimensional map outlined in the second half of Appendix A. The work there only begins to explore the dynamics of the filter map $F_{a,b}$ when a and b assume values drawn from Region 5 of the parameter space. Although the case we studied, namely $b = 0$, is a very specialised instance of the whole range of maps $F_{a,b}$ arising throughout Region 5, nevertheless the phenomenon that we described, that the attracting set splits into increasingly many intervals as a decreases towards 1, is

seen from computer simulations to persist when b is perturbed away from 0. Then, the complicated attractor breaks along its length into isolated components as a approaches 1.

Ideally one would like to explain the dynamics of $F_{a,b}$ for Region 5 via a one-dimensional map even when b is not zero. We point out, though, that the collection of line segments parallel to the contracting direction does not constitute an invariant foliation for $F_{a,b}$; one might suppose a foliation could achieve the desired reduction of dimension when $b \neq 0$, as is possible, for example, with the Plykin map. The discontinuity is responsible for the failure of this approach for $F_{a,b}$; the problem is that a line segment parallel to the stable eigenvector may be split by the discontinuity and the two parts map to different leaves of the foliation.

There is certainly more that could be done to explore the dynamics of the one-dimensional map arising when $b = 0$. Our study concentrated on demonstrating the existence of an attracting set on which the map was topologically transitive, and while this tells us the parts of the interval $[-1, 1]$ visited by orbits, it says nothing about the proportion of the iterates from an orbit that visit each part. One could approach the study from a probabilistic point of view, using ergodic theory, and seek the distribution of iterates across the attracting set.

"Celui qui commence un livre n'est que l'écolier de celui qui l'achève."

— Antoine de la Salle

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Tables

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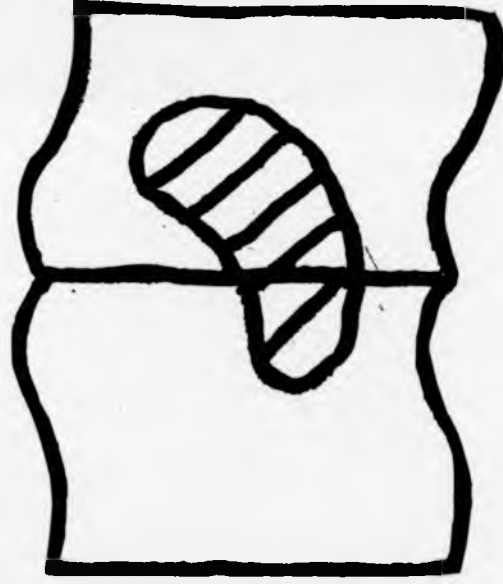


Table 1 Atlas of admissible periodic sequences

The table is a series of lists of admissible periodic sequences, an individual list showing all the sequences of period 1 to 3.00 admissible at a specified parameter value. The parameter values progress from 0.2 to 3.0 in increments of 0.2, and the lists are an extract from the full set of tables at parameter increments of 0.01 compiled by the author. Limitations of space prevent us from reporting here at a finer spacing of the parameter; as a compromise lists at intermediate values have been inserted near the important parameter values $\theta = 0$, $\frac{\pi}{2}$ and π .

Each entry shows a generating string for the sequence together with the interval of admissibility, and where either end-point of the interval coincides with a rational multiple of π , this also is included. Sequences whose generating strings differ only by cycling are not accorded a separate entry.

Close to zero and close to π there are large families of admissible periodic sequences for each of which the generating strings are easily described by the methods of Chapters 7, 8 and 9. Including these families for extremal parameter values would make the lists excessively long; consequently these families have been suppressed when $\theta = 0.05, 0.1, 3.05$ and 3.1 . All other strings are reported as found.

Table 1 Atlas of admissible periodic sequences

Listed up to length 200

Output suppressed for the 1192 sequences of period lengths 1 to 62, whose intervals of admissibility have left end-point zero.

[illegible]

Table 1 Atlas of admissible periodic sequences

Admissible Periodic Sequences and their Intervals of Admissibility : $\theta = 0.1$

(Limited up to length 200)

Output suppressed for the 308 sequences of period lengths 1 to 31, whose intervals of admissibility have left end-point zero.

[illegible][illegible]

Table 1 Atlas of admissible periodic sequences

[illegible]

Table 1 Atlas of admissible periodic sequences

[illegible]

[illegible]

Table 1. Atlas of admissible/periodic sequences

Admissible Periodic Sequences and their Intervals of Admissibility : $\theta = 0.4$

Lasted up to length 200

[illegible]

Table 1 Atlas of admissible periodic sequences

| | |
|-----|--|
| 124 | <p> [3.39966481, 6.40033519] [3.3999981, 6.4000019] [3.4003314, 6.3996686] [3.4006647, 6.3993353] [3.401008, 6.398992] [3.4013413, 6.3986587] [3.4016746, 6.3983254] [3.4020079, 6.3979921] [3.4023412, 6.3976588] [3.4026745, 6.3973255] [3.4030078, 6.3969922] [3.4033411, 6.3966589] [3.4036744, 6.3963256] [3.4040077, 6.3959923] [3.404341, 6.395659] [3.4046743, 6.3953257] [3.4050076, 6.3949924] [3.4053409, 6.3946591] [3.4056742, 6.3943258] [3.4060075, 6.3939925] [3.4063408, 6.3936592] [3.4066741, 6.3933259] [3.4070074, 6.3929926] [3.4073407, 6.3926593] [3.407674, 6.392326] [3.4080073, 6.3919927] [3.4083406, 6.3916594] [3.4086739, 6.3913261] [3.4090072, 6.3909928] [3.4093405, 6.3906595] [3.4096738, 6.3903262] [3.4100071, 6.3899929] [3.4103404, 6.3896596] [3.4106737, 6.3893263] [3.411007, 6.388993] [3.4113403, 6.3886597] [3.4116736, 6.3883264] [3.4120069, 6.3879931] [3.4123402, 6.3876598] [3.4126735, 6.3873265] [3.4130068, 6.3869932] [3.4133401, 6.3866599] [3.4136734, 6.3863266] [3.4140067, 6.3859933] [3.41434, 6.38566] [3.4146733, 6.3853267] [3.4150066, 6.3849934] [3.4153399, 6.3846601] [3.4156732, 6.3843268] [3.4160065, 6.3839935] [3.4163398, 6.3836602] [3.4166731, 6.3833269] [3.4170064, 6.3829936] [3.4173397, 6.3826603] [3.417673, 6.382327] [3.4180063, 6.3819937] [3.4183396, 6.3816604] [3.4186729, 6.3813271] [3.4190062, 6.3809938] [3.4193395, 6.3806605] [3.4196728, 6.3803272] [3.4200061, 6.3799939] [3.4203394, 6.3796606] [3.4206727, 6.3793273] [3.421006, 6.378994] [3.4213393, 6.3786607] [3.4216726, 6.3783274] [3.4220059, 6.3779941] [3.4223392, 6.3776608] [3.4226725, 6.3773275] [3.4230058, 6.3769942] [3.4233391, 6.3766609] [3.4236724, 6.3763276] [3.4240057, 6.3759943] [3.424339, 6.375661] [3.4246723, 6.3753277] [3.4250056, 6.3749944] [3.4253389, 6.3746611] [3.4256722, 6.3743278] [3.4260055, 6.3739945] [3.4263388, 6.3736612] [3.4266721, 6.3733279] [3.4270054, 6.3729946] [3.4273387, 6.3726613] [3.427672, 6.372328] [3.4280053, 6.3719947] [3.4283386, 6.3716614] [3.4286719, 6.3713281] [3.4290052, 6.3709948] [3.4293385, 6.3706615] [3.4296718, 6.3703282] [3.4300051, 6.3699949] [3.4303384, 6.3696616] [3.4306717, 6.3693283] [3.431005, 6.368995] [3.4313383, 6.3686617] [3.4316716, 6.3683284] [3.4320049, 6.3679951] [3.4323382, 6.3676618] [3.4326715, 6.3673285] [3.4330048, 6.3669952] [3.4333381, 6.3666619] [3.4336714, 6.3663286] [3.4340047, 6.3659953] [3.434338, 6.365662] [3.4346713, 6.3653287] [3.4350046, 6.3649954] [3.4353379, 6.3646621] [3.4356712, 6.3643288] [3.4360045, 6.3639955] [3.4363378, 6.3636622] [3.4366711, 6.3633289] [3.4370044, 6.3629956] [3.4373377, 6.3626623] [3.437671, 6.362329] [3.4380043, 6.3619957] [3.4383376, 6.3616624] [3.4386709, 6.3613291] [3.4390042, 6.3609958] [3.4393375, 6.3606625] [3.4396708, 6.3603292] [3.4400041, 6.3599959] [3.4403374, 6.3596626] [3.4406707, 6.3593293] [3.441004, 6.358996] [3.4413373, 6.3586627] [3.4416706, 6.3583294] [3.4420039, 6.3579961] [3.4423372, 6.3576628] [3.4426705, 6.3573295] [3.4430038, 6.3569962] [3.4433371, 6.3566629] [3.4436704, 6.3563296] [3.4440037, 6.3559963] [3.444337, 6.355663] [3.4446703, 6.3553297] [3.4450036, 6.3549964] [3.4453369, 6.3546631] [3.4456702, 6.3543298] [3.4460035, 6.3539965] [3.4463368, 6.3536632] [3.4466701, 6.3533299] [3.4470034, 6.3529966] [3.4473367, 6.3526633] [3.44767, 6.35233] [3.4480033, 6.3519967] [3.4483366, 6.3516634] [3.4486699, 6.3513301] [3.4490032, 6.3509968] [3.4493365, 6.3506635] [3.4496698, 6.3503302] [3.4500031, 6.3499969] [3.4503364, 6.3496636] [3.4506697, 6.3493303] [3.451003, 6.348997] [3.4513363, 6.3486637] [3.4516696, 6.3483304] [3.4520029, 6.3479971] [3.4523362, 6.3476638] [3.4526695, 6.3473305] [3.4530028, 6.3469972] [3.4533361, 6.3466639] [3.4536694, 6.3463306] [3.4540027, 6.3459973] [3.454</p> |
|-----|--|

Table 1 Atlas of admissible periodic sequences

Admissible Periodic Sequences and their Intervals of Admissibility: $\theta = 0.6$

Listed up to length 200

[illegible]

Admissible Periodic Sequences and their Intervals of Admissibility : $\theta = 1.2$

Listed up to length 200

$$\begin{aligned}
1 & \quad 0 \\
2 & \quad 1 \\
3 & \quad \frac{1}{2} \\
4 & \quad \frac{1}{6} \\
5 & \quad \frac{1}{24} \\
6 & \quad \frac{1}{120} \\
7 & \quad \frac{1}{720} \\
8 & \quad \frac{1}{5040} \\
9 & \quad \frac{1}{40320} \\
10 & \quad \frac{1}{362880} \\
11 & \quad \frac{1}{3628800} \\
12 & \quad \frac{1}{39916800} \\
13 & \quad \frac{1}{479001600} \\
14 & \quad \frac{1}{6227020800} \\
15 & \quad \frac{1}{87513600000} \\
16 & \quad \frac{1}{1312761600000} \\
17 & \quad \frac{1}{20995968000000} \\
18 & \quad \frac{1}{355680000000000} \\
19 & \quad \frac{1}{6706080000000000} \\
20 & \quad \frac{1}{134121600000000000} \\
21 & \quad \frac{1}{2816532480000000000} \\
22 & \quad \frac{1}{59542144000000000000} \\
23 & \quad \frac{1}{1279183104000000000000} \\
24 & \quad \frac{1}{27523968000000000000000} \\
25 & \quad \frac{1}{603528192000000000000000} \\
26 & \quad \frac{1}{13679436800000000000000000} \\
27 & \quad \frac{1}{313066240000000000000000000} \\
28 & \quad \frac{1}{7090027520000000000000000000} \\
29 & \quad \frac{1}{160000000000000000000000000000} \\
30 & \quad \frac{1}{3603600000000000000000000000000} \\
31 & \quad \frac{1}{81081000000000000000000000000000} \\
32 & \quad \frac{1}{182330400000000000000000000000000} \\
33 & \quad \frac{1}{4112280000000000000000000000000000} \\
34 & \quad \frac{1}{92470080000000000000000000000000000} \\
35 & \quad \frac{1}{2054400000000000000000000000000000000} \\
36 & \quad \frac{1}{46200000000000000000000000000000000000} \\
37 & \quad \frac{1}{102960000000000000000000000000000000000} \\
38 & \quad \frac{1}{230592000000000000000000000000000000000} \\
39 & \quad \frac{1}{5143680000000000000000000000000000000000} \\
40 & \quad \frac{1}{11520000000000000000000000000000000000000} \\
41 & \quad \frac{1}{25804800000000000000000000000000000000000} \\
42 & \quad \frac{1}{576576000000000000000000000000000000000000} \\
43 & \quad \frac{1}{1296144000000000000000000000000000000000000} \\
44 & \quad \frac{1}{29163200000000000000000000000000000000000000} \\
45 & \quad \frac{1}{656172800000000000000000000000000000000000000} \\
46 & \quad \frac{1}{1474608000000000000000000000000000000000000000} \\
47 & \quad \frac{1}{332121600} \\
48 & \quad \frac{1}{74649600} \\
49 & \quad \frac{1}{167961600} \\
50 & \quad \frac{1}{377971200} \\
51 & \quad \frac{1}{8519616000} \\
52 & \quad \frac{1}{19124064000} \\
53 & \quad \frac{1}{4300672000} \\
54 & \quad \frac{1}{96768000} \\
55 & \quad \frac{1}{2167200} \\
56 & \quad \frac{1}{48768000} \\
57 & \quad \frac{1}{109344000} \\
58 & \quad \frac{1}{2428800} \\
59 & \quad \frac{1}{54528000} \\
60 & \quad \frac{1}{1219200} \\
61 & \quad \frac{1}{2745600} \\
62 & \quad \frac{1}{614400} \\
63 & \quad \frac{1}{1392000} \\
64 & \quad \frac{1}{3110400} \\
65 & \quad \frac{1}{6998400} \\
66 & \quad \frac{1}{15724800} \\
67 & \quad \frac{1}{35430400} \\
68 & \quad \frac{1}{79264000} \\
69 & \quad \frac{1}{177472000} \\
70 & \quad \frac{1}{398016000} \\
71 & \quad \frac{1}{895872000} \\
72 & \quad \frac{1}{19993600} \\
73 & \quad \frac{1}{44985600} \\
74 & \quad \frac{1}{101184000} \\
75 & \quad \frac{1}{227008000} \\
76 & \quad \frac{1}{509824000} \\
77 & \quad \frac{1}{1146496000} \\
78 & \quad \frac{1}{25804800} \\
79 & \quad \frac{1}{57657600} \\
80$$
Admissible Periodic Sequences and their Intervals of Admissibility : $\theta = 1.4$

Listed up to length 200

```

1  1  (6.1490615e-07,3.4189214e+0,0.0)
2  1  +
3  2  (6.1490615e-07,1.5797963e+0,(0.4)
4  2  -
5  3  (1.0819795e+0,1.5797963e+0,(0.4)
6  3  -
7  4  (1.5497963e+0,1.5797963e+0,(0.4)
8  4  -
9  5  (1.5497963e+0,1.5797963e+0,(0.4)
10 5  -
11 6  (1.5497963e+0,1.5797963e+0,(0.4)
12 6  -
13 7  (1.5497963e+0,1.5797963e+0,(0.4)
14 7  -
15 8  (1.5497963e+0,1.5797963e+0,(0.4)
16 8  -
17 9  (1.5497963e+0,1.5797963e+0,(0.4)
18 9  -
19 10 (1.5497963e+0,1.5797963e+0,(0.4)
20 10 -
21 11 (1.5497963e+0,1.5797963e+0,(0.4)
22 11 -
23 12 (1.5497963e+0,1.5797963e+0,(0.4)
24 12 -
25 13 (1.5497963e+0,1.5797963e+0,(0.4)
26 13 -
27 14 (1.5497963e+0,1.5797963e+0,(0.4)
28 14 -
29 15 (1.5497963e+0,1.5797963e+0,(0.4)
30 15 -
31 16 (1.5497963e+0,1.5797963e+0,(0.4)
32 16 -
33 17 (1.5497963e+0,1.5797963e+0,(0.4)
34 17 -
35 18 (1.5497963e+0,1.5797963e+0,(0.4)
36 18 -
37 19 (1.5497963e+0,1.5797963e+0,(0.4)
38 19 -
39 20 (1.5497963e+0,1.5797963e+0,(0.4)
40 20 -
41 21 (1.5497963e+0,1.5797963e+0,(0.4)
42 21 -
43 22 (1.5497963e+0,1.5797963e+0,(0.4)
44 22 -
45 23 (1.5497963e+0,1.5797963e+0,(0.4)
46 23 -
47 24 (1.5497963e+0,1.5797963e+0,(0.4)
48 24 -
49 25 (1.5497963e+0,1.5797963e+0,(0.4)
50 25 -
51 26 (1.5497963e+0,1.5797963e+0,(0.4)
52 26 -
53 27 (1.5497963e+0,1.5797963e+0,(0.4)
54 27 -
55 28 (1.5497963e+0,1.5797963e+0,(0.4)
56 28 -
57 29 (1.5497963e+0,1.5797963e+0,(0.4)
58 29 -
59 30 (1.5497963e+0,1.5797963e+0,(0.4)
60 30 -
61 31 (1.5497963e+0,1.5797963e+0,(0.4)
62 31 -
63 32 (1.5497963e+0,1.5797963e+0,(0.4)
64 32 -
65 33 (1.5497963e+0,1.5797963e+0,(0.4)
66 33 -
67 34 (1.5497963e+0,1.5797963e+0,(0.4)
68 34 -
69 35 (1.5497963e+0,1.5797963e+0,(0.4)
70 35 -
71 36 (1.5497963e+0,1.5797963e+0,(0.4)
72 36 -
73 37 (1.5497963e+0,1.5797963e+0,(0.4)
74 37 -
75 38 (1.5497963e+0,1.5797963e+0,(0.4)
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86 43 -
87 44 (1.5497963e+0,1.5797963e+0,(0.4)
88 44 -
89 45 (1.5497963e+0,1.5797963e+0,(0.4)
90 45 -
91 46 (1.5497963e+0,1.5797963e+0,(0.4)
92 46 -
93 47 (1.5497963e+0,1.5797963e+0,(0.4)
94 47 -
95 48 (1.5497963e+0,1.5797963e+0,(0.4)
96 48 -
97 49 (1.5497963e+0,1.5797963e+0,(0.4)
98 49 -
99 50 (1.5497963e+0,1.5797963e+0,(0.4)
100 50 -
101 51 (1.5497963e+0,1.5797963e+0,(0.4)
102 51 -
103 52 (1.5497963e+0,1.5797963e+0,(0.4)
104 52 -
105 53 (1.5497963e+0,1.5797963e+0,(0.4)
106 53 -
107 54 (1.5497963e+0,1.5797963e+0,(0.4)
108 54 -
109 55 (1.5497963e+0,1.5797963e+0,(0.4)
110 55 -
111 56 (1.5497963e+0,1.5797963e+0,(0.4)
112 56 -
113 57 (1.5497963e+0,1.5797963e+0,(0.4)
114 57 -
115 58 (1.5497963e+0,1.5797963e+0,(0.4)
116 58 -
117 59 (1.5497963e+0,1.5797963e+0,(0.4)
118 59 -
119 60 (1.5497963e+0,1.5797963e+0,(0.4)
120 60 -
121 61 (1.5497963e+0,1.5797963e+0,(0.4)
122 61 -
123 62 (1.5497963e+0,1.5797963e+0,(0.4)
124 62 -
125 63 (1.5497963e+0,1.5797963e+0,(0.4)
126 63 -
127 64 (1.5497963e+0,1.5797963e+0,(0.4)
128 64 -
129 65 (1.5497963e+0,1.5797963e+0,(0.4)
130 65 -
131 66 (1.5497963e+0,1.5797963e+0,(0.4)
132 66 -
133 67 (1.5497963e+0,1.5797963e+0,(0.4)
134 67 -
135 68 (1.5497963e+0,1.5797963e+0,(0.4)
136 68 -
137 69 (1.5497963e+0,1.5797963e+0,(0.4)
138 69 -
139 70 (1.5497963e+0,1.5797963e+0,(0.4)
140 70 -
141 71 (1.5497963e+0,1.5797963e+0,(0.4)
142 71 -
143 72 (1.5497963e+0,1.5797963e+0,(0.4)
144 72 -
145 73 (1.5497963e+0,1.5797963e+0,(0.4)
146 73 -
147 74 (1.5497963e+0,1.5797963e+0,(0.4)
148 74 -
149 75 (1.5497963e+0,1.5797963e+0,(0.4)
150 75 -
151 76 (1.5497963e+0,1.5797963e+0,(0.4)
152 76 -
153 77 (1.5497963e+0,1.5797963e+0,(0.4)
154 77 -
155 78 (1.5497963e+0,1.5797963e+0,(0.4)
156 78 -
157 79 (1.5497963e+0,1.5797963e+0,(0.4)
158 79 -
159 80 (1.5497963e+0,1.5797963e+0,(0.4)
160 80 -
161 81 (1.5497963e+0,1.5797963e+0,(0.4)
162 81 -
163 82 (1.5497963e+0,1.5797963e+0,(0.4)
164 82 -
165 83 (1.5497963e+0,1.5797963e+0,(0.4)
166 83 -
167 84 (1.5497963e+0,1.5797963e+0,(0.4)
168 84 -
169 85 (1.5497963e+0,1.5797963e+0,(0.4)
170 85 -
171 86 (1.5497963e+0,1.5797963e+0,(0.4)
172 86 -
173 87 (1.5497963e+0,1.5797963e+0,(0.4)
174 87 -
175 88 (1.5497963e+0,1.5797963e+0,(0.4)
176 88 -
177 89 (1.5497963e+0,1.5797963e+0,(0.4)
178 89 -
179 90 (1.5497963e+0,1.5797963e+0,(0.4)
180 90 -
181 91 (1.5497963e+0,1.5797963e+0,(0.4)
182 91 -
183 
```

Admissible Periodic Sequences and their Intervals of Admissibility : $\theta = 1.55$

Listed up to length 200

| | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|-----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | 81 | 82 | 83 | 84 | 85 | 86 | 87 | 88 | 89 | 90 | 91 | 92 | 93 | 94 | 95 | 96 | 97 | 98 | 99 | 100 |
| 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 | 36 | 37 | 38 | 39 | 40 | 41 | 42 | 43 | 44 | 45 | 46 | 47 | 48 | 49 | 50 | 51 | 52 | 53 | 54 | 55 | 56 | 57 | 58 | 59 | 60 | 61 | 62 | 63 | 64 | 65 | 66 | 67 | 68 | 69 | 70 | 71 | 72 | 73 | 74 | 75 | 76 | 77 | 78 | 79 | 80 | | | | | | | | | | | | | | | | | | | | |

$$(1.53777513, 1.57079633) = \left(\frac{29}{19}, \frac{9}{5} \right)$$
[illegible]

[illegible]Admissible Periodic Sequences and their Intervals of Admissibility: $\theta = 1.6$

Listed up to length 200

[illegible][illegible][illegible]

Admissible Periodic Sequences and their Intervals of Admissibility : $\theta = 1.8$

Listed up to length 200

[illegible]

Admissible Periodic Sequences and their Intervals of Admissibility : $\theta = 2$

Listed up to length 200

[illegible]Admissible Periodic Sequences and their Intervals of Admissibility : $\theta = 2.2$

Listed up to length 200

[illegible]

Admissible Periodic Sequences and their Intervals of Admissibility: $\theta = 2.4$

Listed up to length 200

[illegible]

Admissible Periodic Sequences and their Intervals of Admissibility : $\theta = 2.6$

Listed up to length 300

[illegible]

Table 1 Atlas of admissible periodic sequences

[illegible]

Table 1 Atlas of admissible periodic sequences

Admissible Periodic Sequences and their Intervals of Admissibility : $\theta = 3$

I listed him to benefit 2000

$$\begin{aligned}
3 : & \quad (1.5792681, 3.14159211) = \left(\frac{3}{2}, \pi\right) \\
4 : & \quad (1.5792681, 3.14159211) = \left(\frac{3}{2}, \pi\right) \\
5 : & \quad (6.782125 - 47.31439211i) = (-47, 6.782125) \\
6 : & \quad (2.5501646, 3.14159211) = \left(\frac{25501646}{10^9}, \pi\right) \\
7 : & \quad 500+ \\
8 : & \quad +50050+ \\
9 : & \quad (2.6179938, 3.14159211) = \left(\frac{26179938}{10^6}, \pi\right) \\
10 : & \quad (2.6179938, 3.14159211) = \left(\frac{26179938}{10^6}, \pi\right) \\
11 : & \quad -8+4+0 \\
12 : & \quad 0+ \\
13 : & \quad 0+ \\
14 : & \quad 0+ \\
15 : & \quad 0+ \\
16 : & \quad 0+ \\
17 : & \quad 0+ \\
18 : & \quad 0+ \\
19 : & \quad 0+ \\
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95 : & \quad 0+ \\
96 : & \quad 0+ \\
97 : & \quad 0+ \\
98 : & \quad 0+ \\
99 : & \quad 0+ \\
100 : & \quad 0+
\end{aligned}$$

Table 1 Atlas of admissible periodic sequences

Admissible Periodic Sequences and their Intervals of Admissibility ; $\theta = 2.8$

Listed up to length 200

[illegible]

Admissible Periodic Sequences and their Intervals of Admissibility: $\theta = 3.1$

Listed up to length 200

All 1155 sequences suppressed: each sequence found when $\theta = 3.1$ with period length up to 200 is one whose interval of admissibility has right end-point π .

Table 1 Atlas of admissible periodic sequences

Table 2 Count of sequences found at uniformly spaced parameter values

Tabulated for parameter values advancing from 0 to π by increments of 0.01 are the counts of admissible periodic sequences of period length 1 to 200 found by computer searches. These numbers exhibit the incidence of sequences at parameter values which could not be reported in Table 1.

Table 2 Count of sequences found at uniformly spaced parameter values

| θ | Count | θ | Count | θ | Count | θ | Count | θ | Count |
|----------|-------|----------|-------|----------|-------|----------|-------|----------|-------|
| 0.01 | 12337 | 0.41 | 33 | 0.81 | 24 | 1.21 | 34 | 1.61 | 26 |
| 0.02 | 7560 | 0.42 | 28 | 0.82 | 18 | 1.22 | 36 | 1.62 | 25 |
| 0.03 | 3326 | 0.43 | 23 | 0.83 | 12 | 1.23 | 40 | 1.63 | 43 |
| 0.04 | 1350 | 0.44 | 24 | 0.84 | 22 | 1.24 | 26 | 1.64 | 35 |
| 0.05 | 506 | 0.45 | 22 | 0.85 | 17 | 1.25 | 38 | 1.65 | 30 |
| 0.06 | 906 | 0.46 | 32 | 0.86 | 18 | 1.26 | 19 | 1.66 | 31 |
| 0.07 | 684 | 0.47 | 32 | 0.87 | 16 | 1.27 | 21 | 1.67 | 27 |
| 0.08 | 592 | 0.48 | 36 | 0.88 | 22 | 1.28 | 29 | 1.68 | 29 |
| 0.09 | 478 | 0.49 | 34 | 0.89 | 22 | 1.29 | 26 | 1.69 | 22 |
| 0.1 | 410 | 0.5 | 23 | 0.9 | 19 | 1.3 | 30 | 1.7 | 30 |
| 0.11 | 340 | 0.51 | 24 | 0.91 | 27 | 1.31 | 30 | 1.71 | 27 |
| 0.12 | 328 | 0.52 | 19 | 0.92 | 24 | 1.32 | 20 | 1.72 | 29 |
| 0.13 | 282 | 0.53 | 32 | 0.93 | 32 | 1.33 | 18 | 1.73 | 43 |
| 0.14 | 229 | 0.54 | 25 | 0.94 | 22 | 1.34 | 21 | 1.74 | 38 |
| 0.15 | 220 | 0.55 | 22 | 0.95 | 22 | 1.35 | 27 | 1.75 | 25 |
| 0.16 | 104 | 0.56 | 23 | 0.96 | 23 | 1.36 | 33 | 1.76 | 31 |
| 0.17 | 185 | 0.57 | 16 | 0.97 | 16 | 1.37 | 27 | 1.77 | 40 |
| 0.18 | 170 | 0.58 | 22 | 0.98 | 22 | 1.38 | 28 | 1.78 | 34 |
| 0.19 | 153 | 0.59 | 19 | 0.99 | 19 | 1.39 | 27 | 1.79 | 31 |
| 0.2 | 161 | 0.6 | 27 | 1 | 14 | 1.4 | 28 | 1.8 | 32 |
| 0.21 | 124 | 0.61 | 24 | 1.01 | 24 | 1.41 | 42 | 2.01 | 28 |
| 0.22 | 132 | 0.62 | 18 | 1.02 | 18 | 1.42 | 44 | 2.02 | 30 |
| 0.23 | 111 | 0.63 | 22 | 1.03 | 22 | 1.43 | 35 | 2.03 | 35 |
| 0.24 | 112 | 0.64 | 30 | 1.04 | 30 | 1.44 | 34 | 2.04 | 45 |
| 0.25 | 96 | 0.65 | 17 | 1.05 | 17 | 1.45 | 40 | 2.05 | 46 |
| 0.26 | 102 | 0.66 | 18 | 1.06 | 18 | 1.46 | 35 | 2.06 | 57 |
| 0.27 | 104 | 0.67 | 23 | 1.07 | 23 | 1.47 | 41 | 2.07 | 66 |
| 0.28 | 93 | 0.68 | 25 | 1.08 | 25 | 1.48 | 48 | 2.08 | 71 |
| 0.29 | 100 | 0.69 | 30 | 1.09 | 30 | 1.49 | 44 | 2.09 | 81 |
| 0.3 | 76 | 0.7 | 27 | 1.1 | 43 | 1.5 | 45 | 2.1 | 48 |
| 0.31 | 77 | 0.71 | 21 | 1.01 | 42 | 1.51 | 45 | 2.11 | 65 |
| 0.32 | 77 | 0.72 | 25 | 1.02 | 25 | 1.52 | 50 | 2.12 | 72 |
| 0.33 | 72 | 0.73 | 22 | 1.03 | 22 | 1.53 | 41 | 2.13 | 81 |
| 0.34 | 78 | 0.74 | 21 | 1.04 | 21 | 1.54 | 64 | 2.14 | 47 |
| 0.35 | 58 | 0.75 | 28 | 1.05 | 28 | 1.55 | 75 | 2.15 | 45 |
| 0.36 | 50 | 0.76 | 30 | 1.06 | 30 | 1.56 | 119 | 2.16 | 38 |
| 0.37 | 56 | 0.77 | 24 | 1.07 | 24 | 1.57 | 165 | 2.17 | 28 |
| 0.38 | 65 | 0.78 | 30 | 1.08 | 30 | 1.58 | 133 | 2.18 | 35 |
| 0.39 | 41 | 0.79 | 41 | 1.09 | 41 | 1.59 | 13 | 2.19 | 25 |
| 0.4 | 37 | 0.8 | 41 | 1.1 | 46 | 1.6 | 68 | 2.2 | 26 |
| 0.41 | 63 | 1.01 | 42 | 1.61 | 54 | 2.21 | 26 | | |
| 0.42 | 38 | 1.02 | 65 | 1.62 | 46 | 2.22 | 33 | | |
| 0.43 | 42 | 1.03 | 52 | 1.63 | 40 | 2.23 | 32 | | |
| 0.44 | 52 | 1.04 | 77 | 1.64 | 54 | 2.24 | 27 | | |
| 0.45 | 34 | 1.05 | 86 | 1.65 | 43 | 2.25 | 28 | | |
| 0.46 | 38 | 1.06 | 76 | 1.66 | 62 | 2.26 | 34 | | |
| 0.47 | 32 | 1.07 | 61 | 1.67 | 56 | 2.27 | 28 | | |
| 0.48 | 37 | 1.08 | 57 | 1.68 | 56 | 2.28 | 30 | | |
| 0.49 | 42 | 1.09 | 36 | 1.69 | 38 | 2.29 | 30 | | |
| 0.5 | 44 | 1.1 | 46 | 1.7 | 43 | 2.3 | 38 | | |
| 0.51 | 41 | 1.11 | 33 | 1.71 | 39 | 2.31 | 36 | | |
| 0.52 | 46 | 1.12 | 26 | 1.72 | 44 | 2.32 | 45 | | |
| 0.53 | 51 | 1.13 | 32 | 1.73 | 45 | 2.33 | 20 | | |
| 0.54 | 25 | 1.14 | 34 | 1.74 | 50 | 2.34 | 20 | | |
| 0.55 | 31 | 1.15 | 22 | 1.75 | 41 | 2.35 | 24 | | |
| 0.56 | 27 | 1.16 | 29 | 1.76 | 36 | 2.36 | 23 | | |
| 0.57 | 44 | 1.17 | 30 | 1.77 | 32 | 2.37 | 19 | | |
| 0.58 | 43 | 1.18 | 23 | 1.78 | 36 | 2.38 | 45 | | |
| 0.59 | 48 | 1.19 | 22 | 1.79 | 28 | 2.39 | 24 | | |
| 0.6 | 33 | 1.2 | 27 | 1.8 | 29 | 2.4 | 19 | | |

Table 2 Count of sequences found at uniformly spaced parameter values

| θ | Count | θ | Count | θ | Count | θ | Count |
|----------|-------|----------|-------|----------|-------|----------|-------|
| 2.41 | 27 | 2.61 | 30 | 2.81 | 54 | 3.01 | 165 |
| 2.42 | 30 | 2.62 | 29 | 2.82 | 58 | 3.02 | 211 |
| 2.43 | 27 | 2.63 | 56 | 2.83 | 60 | 3.03 | 233 |
| 2.44 | 36 | 2.64 | 48 | 2.84 | 48 | 3.04 | 271 |
| 2.45 | 38 | 2.65 | 41 | 2.85 | 68 | 3.05 | 321 |
| 2.46 | 29 | 2.66 | 42 | 2.86 | 71 | 3.06 | 355 |
| 2.47 | 36 | 2.67 | 31 | 2.87 | 67 | 3.07 | 436 |
| 2.48 | 20 | 2.68 | 33 | 2.88 | 62 | 3.08 | 602 |
| 2.49 | 23 | 2.69 | 35 | 2.89 | 74 | 3.09 | 779 |
| 2.5 | 26 | 2.7 | 40 | 2.9 | 72 | 3.1 | 1155* |
| 2.51 | 17 | 2.71 | 47 | 2.91 | 68 | 3.11 | 2002* |
| 2.52 | 25 | 2.72 | 29 | 2.92 | 93 | 3.12 | 3128* |
| 2.53 | 30 | 2.73 | 41 | 2.93 | 95 | 3.13 | 5086* |
| 2.54 | 36 | 2.74 | 49 | 2.94 | 103 | 3.14 | 5886* |
| 2.55 | 34 | 2.75 | 46 | 2.95 | 109 | | |
| 2.56 | 23 | 2.76 | 59 | 2.96 | 121 | | |
| 2.57 | 26 | 2.77 | 44 | 2.97 | 141 | | |
| 2.58 | 27 | 2.78 | 39 | 2.98 | 128 | | |
| 2.59 | 30 | 2.79 | 45 | 2.99 | 145 | | |
| 2.6 | 35 | 2.8 | 48 | 3 | 172 | | |

*For this parameter value all sequences found are of the type investigated in Chapters 7, 8 and 9 and there referred to as "admissible adjacent to zero"; thus the count coincides with

$$\sum_{i=1}^m \psi(r_i)$$

where $\psi(r)$ is the Euler totient function.

*For this parameter value all sequences found are accounted for by sequences admissible adjacent to zero under the transformation that requires alternate digits. The corresponding formula for the count is

$$3 + \sum_{i=1}^m \left(\left\lfloor \frac{r_i}{2} \right\rfloor - 1 \right) \psi(r_i) + \frac{1}{2} \sum_{i=1}^m \left(\left\lfloor \frac{r_i}{2} \right\rfloor + 1 \right) \psi(r_i)$$

The table lists all strings of length not exceeding 20, that generate a periodic sequence admissible throughout some open interval for the parameter θ .

As well as the interval of admissibility, a column in the table is provided for the point within the interval where the divergence between the f_i 's and g_i the functions of the Chua-Lin inequalities, attains its maximum, i.e. the maximum point of

$$\min_{0 \leq i \leq n-1} \{ |g(\theta) - f_i(\theta)|, |g(\theta) + f_i(\theta)| \}.$$

This value is included only when it coincides with a rational multiple of π .

In this table, negated versions of the strings have been suppressed.

Table 3 Comprehensive list of admissible periodic sequences for period length not exceeding 20

| Length | String | Interval of Admissibility | Max Point |
|--------|---|--------------------------------|-----------------------------------|
| 1 | 0 | (5.507385324, -07, 3.1415921) | $=(0, \pi)$ |
| 2 | +- | (1.57079633, 3.1415921) | $=(\frac{\pi}{2}, \pi)$ |
| 3 | +-0 | (6.74349576, -07, 1.57079633) | $=(0, \frac{\pi}{2})$ |
| 4 | 00-- | (6.6222289, -07, 1.04719755) | $=(0, \frac{\pi}{2})$ |
| 5 | 0000-- | (1.04719755, 1.57079633) | $=(\frac{\pi}{2}, \frac{\pi}{2})$ |
| 6 | 000000-- | (2.3540449, 3.14159215) | $=(\frac{\pi}{2}, \pi)$ |
| 7 | 00000000-- | (5.04030864, -07, 0.785398163) | $=(0, \frac{\pi}{2})$ |
| 8 | 0000000000-- | (5.74033256, -07, 0.628318531) | $=(0, \frac{\pi}{2})$ |
| 9 | 000000000000-- | (5.74033256, -07, 0.628318531) | $=(0, \frac{\pi}{2})$ |
| 10 | 00000000000000-- | (1.57079633, 1.84465559) | $=(\frac{\pi}{2}, \frac{\pi}{2})$ |
| 11 | 0000000000000000-- | (6.78943736, -07, 0.323598778) | $=(0, \frac{\pi}{2})$ |
| 12 | 000000000000000000-- | (1.57079633, 2.0943951) | $=(\frac{\pi}{2}, \frac{\pi}{2})$ |
| 13 | 00000000000000000000-- | (2.0943951, 3.14159212) | $=(\frac{\pi}{2}, \pi)$ |
| 14 | 0000000000000000000000-- | (2.61799388, 3.14159197) | $=(\frac{\pi}{2}, \pi)$ |
| 15 | 000000000000000000000000-- | (5.83242847, -07, 0.445798951) | $=(0, \frac{\pi}{2})$ |
| 16 | 00000000000000000000000000-- | (5.83242847, -07, 0.445798951) | $=(0, \frac{\pi}{2})$ |
| 17 | 0000000000000000000000000000-- | (5.83242847, -07, 0.445798951) | $=(0, \frac{\pi}{2})$ |
| 18 | 000000000000000000000000000000-- | (1.34639685, 1.57079633) | $=(\frac{\pi}{2}, \frac{\pi}{2})$ |
| 19 | 00000000000000000000000000000000-- | (5.11033814, -07, 0.392699082) | $=(0, \frac{\pi}{2})$ |
| 20 | 0000000000000000000000000000000000-- | (5.11033814, -07, 0.392699082) | $=(0, \frac{\pi}{2})$ |
| 21 | 000000000000000000000000000000000000-- | (2.74889357, 3.14159214) | $=(\frac{\pi}{2}, \pi)$ |
| 22 | 00000000000000000000000000000000000000-- | (2.74889357, 3.14159214) | $=(\frac{\pi}{2}, \pi)$ |
| 23 | 00-- | (6.42998487, -07, 0.34906585) | $=(0, \frac{\pi}{2})$ |
| 24 | 00-- | (6.42998487, -07, 0.34906585) | $=(0, \frac{\pi}{2})$ |
| 25 | 00-- | (0.915801896, 1.04719755) | $=(0, \frac{\pi}{2})$ |
| 26 | 00-- | (1.04719755, 1.5822212) | $=(\frac{\pi}{2}, \pi)$ |
| 27 | 00-- | (1.57079633, 1.74532925) | $=(\frac{\pi}{2}, \pi)$ |
| 28 | 00-- | (1.74532925, 1.98337054) | $=(\frac{\pi}{2}, \pi)$ |
| 29 | 00-- | (2.2579076, 2.4436695) | $=(\frac{\pi}{2}, \pi)$ |
| 30 | 00-- | (5.79231714, -07, 0.31459265) | $=(0, \frac{\pi}{2})$ |
| 31 | 00-- | (5.79231714, -07, 0.31459265) | $=(0, \frac{\pi}{2})$ |
| 32 | 00-- | (1.25663706, 1.57079633) | $=(\frac{\pi}{2}, \pi)$ |
| 33 | 00-- | (2.51327413, 3.141592) | $=(\frac{\pi}{2}, \pi)$ |
| 34 | 00-- | (2.51327413, 3.141592) | $=(\frac{\pi}{2}, \pi)$ |
| 35 | 00-- | (2.82743339, 3.14159207) | $=(\frac{\pi}{2}, \pi)$ |
| 36 | 00-- | (2.82743339, 3.14159207) | $=(\frac{\pi}{2}, \pi)$ |
| 37 | 00-- | (5.26835666, -07, 0.265599332) | $=(0, \frac{\pi}{2})$ |
| 38 | 00-- | (5.26835666, -07, 0.265599332) | $=(0, \frac{\pi}{2})$ |
| 39 | 00-- | (5.26835666, -07, 0.265599332) | $=(0, \frac{\pi}{2})$ |
| 40 | 00-- | (5.26835666, -07, 0.265599332) | $=(0, \frac{\pi}{2})$ |
| 41 | 00-- | (1.42799666, 1.57079633) | $=(\frac{\pi}{2}, \pi)$ |
| 42 | 00-- | (6.83182584, -07, 0.261799388) | $=(0, \frac{\pi}{2})$ |
| 43 | 00-- | (6.83182584, -07, 0.261799388) | $=(0, \frac{\pi}{2})$ |
| 44 | 000-- | (1.21075578, 1.28751985) | $=(0, \frac{\pi}{2})$ |
| 45 | 000-- | (1.8540728, 1.93083688) | $=(0, \frac{\pi}{2})$ |
| 46 | 000-- | (2.87979327, 3.14159197) | $=(\frac{\pi}{2}, \pi)$ |

Table 2 Count of sequences found at uniformly spaced parameter values

Table 3 Comprehensive list of admissible periodic sequences for period length not exceeding 20

| Length | Interval of Admissibility | Functions | Length | Interval of Admissibility | Functions |
|--------|---------------------------|--------------------|--------|---------------------------|--------------------|
| 500 | (1.38531333, 1.280218615) | $\Psi_{500}^{(1)}$ | 678 | (1.78236661, 1.78236671) | $\Psi_{678}^{(1)}$ |
| 501 | (1.38483731, 1.280038811) | $\Psi_{501}^{(1)}$ | 680 | (1.78236661, 1.78236671) | $\Psi_{680}^{(1)}$ |
| 502 | (1.38436129, 1.279867050) | $\Psi_{502}^{(1)}$ | 682 | (1.78236672, 1.78236682) | $\Psi_{682}^{(1)}$ |
| 503 | (1.38388526, 1.279695290) | $\Psi_{503}^{(1)}$ | 684 | (1.78236673, 1.78236683) | $\Psi_{684}^{(1)}$ |
| 504 | (1.38340924, 1.279523530) | $\Psi_{504}^{(1)}$ | 686 | (1.78236674, 1.78236684) | $\Psi_{686}^{(1)}$ |
| 505 | (1.38293321, 1.279351770) | $\Psi_{505}^{(1)}$ | 688 | (1.78236675, 1.78236685) | $\Psi_{688}^{(1)}$ |
| 506 | (1.38245719, 1.279179990) | $\Psi_{506}^{(1)}$ | 690 | (1.78236676, 1.78236686) | $\Psi_{690}^{(1)}$ |
| 507 | (1.38198116, 1.279008230) | $\Psi_{507}^{(1)}$ | 692 | (1.78236677, 1.78236687) | $\Psi_{692}^{(1)}$ |
| 508 | (1.38150514, 1.278836470) | $\Psi_{508}^{(1)}$ | 694 | (1.78236678, 1.78236688) | $\Psi_{694}^{(1)}$ |
| 509 | (1.38102911, 1.278664710) | $\Psi_{509}^{(1)}$ | 696 | (1.78236679, 1.78236689) | $\Psi_{696}^{(1)}$ |
| 510 | (1.38055309, 1.278492950) | $\Psi_{510}^{(1)}$ | 698 | (1.78236680, 1.78236690) | $\Psi_{698}^{(1)}$ |
| 511 | (1.38007706, 1.278321190) | $\Psi_{511}^{(1)}$ | 700 | (1.78236681, 1.78236691) | $\Psi_{700}^{(1)}$ |
| 512 | (1.37960104, 1.278149430) | $\Psi_{512}^{(1)}$ | 702 | (1.78236682, 1.78236692) | $\Psi_{702}^{(1)}$ |
| 513 | (1.37912501, 1.277977670) | $\Psi_{513}^{(1)}$ | 704 | (1.78236683, 1.78236693) | $\Psi_{704}^{(1)}$ |
| 514 | (1.37864899, 1.277805910) | $\Psi_{514}^{(1)}$ | 706 | (1.78236684, 1.78236694) | $\Psi_{706}^{(1)}$ |
| 515 | (1.37817296, 1.277634150) | $\Psi_{515}^{(1)}$ | 708 | (1.78236685, 1.78236695) | $\Psi_{708}^{(1)}$ |
| 516 | (1.37769694, 1.277462390) | $\Psi_{516}^{(1)}$ | 710 | (1.78236686, 1.78236696) | $\Psi_{710}^{(1)}$ |
| 517 | (1.37722091, 1.277290630) | $\Psi_{517}^{(1)}$ | 712 | (1.78236687, 1.78236697) | $\Psi_{712}^{(1)}$ |
| 518 | (1.37674489, 1.277118870) | $\Psi_{518}^{(1)}$ | 714 | (1.78236688, 1.78236698) | $\Psi_{714}^{(1)}$ |
| 519 | (1.37626886, 1.276947110) | $\Psi_{519}^{(1)}$ | 716 | (1.78236689, 1.78236699) | $\Psi_{716}^{(1)}$ |
| 520 | (1.37579284, 1.276775350) | $\Psi_{520}^{(1)}$ | 718 | (1.78236690, 1.78236700) | $\Psi_{718}^{(1)}$ |
| 521 | (1.37531681, 1.276603590) | $\Psi_{521}^{(1)}$ | 720 | (1.78236691, 1.78236701) | $\Psi_{720}^{(1)}$ |
| 522 | (1.37484079, 1.276431830) | $\Psi_{522}^{(1)}$ | 722 | (1.78236692, 1.78236702) | $\Psi_{722}^{(1)}$ |
| 523 | (1.37436476, 1.276260070) | $\Psi_{523}^{(1)}$ | 724 | (1.78236693, 1.78236703) | $\Psi_{724}^{(1)}$ |
| 524 | (1.37388874, 1.276088310) | $\Psi_{524}^{(1)}$ | 726 | (1.78236694, 1.78236704) | $\Psi_{726}^{(1)}$ |
| 525 | (1.37341271, 1.275916550) | $\Psi_{525}^{(1)}$ | 728 | (1.78236695, 1.78236705) | $\Psi_{728}^{(1)}$ |
| 526 | (1.37293669, 1.275744790) | $\Psi_{526}^{(1)}$ | 730 | (1.78236696, 1.78236706) | $\Psi_{730}^{(1)}$ |
| 527 | (1.37246066, 1.275573030) | $\Psi_{527}^{(1)}$ | 732 | (1.78236697, 1.78236707) | $\Psi_{732}^{(1)}$ |
| 528 | (1.37198464, 1.275401270) | $\Psi_{528}^{(1)}$ | 734 | (1.78236698, 1.78236708) | $\Psi_{734}^{(1)}$ |
| 529 | (1.37150861, 1.275229510) | $\Psi_{529}^{(1)}$ | 736 | (1.78236699, 1.78236709) | $\Psi_{736}^{(1)}$ |
| 530 | (1.37103259, 1.275057750) | $\Psi_{530}^{(1)}$ | 738 | (1.78236700, 1.78236710) | $\Psi_{738}^{(1)}$ |
| 531 | (1.37055656, 1.274886000) | $\Psi_{531}^{(1)}$ | 740 | (1.78236701, 1.78236711) | $\Psi_{740}^{(1)}$ |
| 532 | (1.37008054, 1.274714240) | $\Psi_{532}^{(1)}$ | 742 | (1.78236702, 1.78236712) | $\Psi_{742}^{(1)}$ |
| 533 | (1.36960451, 1.274542480) | $\Psi_{533}^{(1)}$ | 744 | (1.78236703, 1.78236713) | $\Psi_{744}^{(1)}$ |
| 534 | (1.36912849, 1.274370720) | $\Psi_{534}^{(1)}$ | 746 | (1.78236704, 1.78236714) | $\Psi_{746}^{(1)}$ |
| 535 | (1.36865246, 1.274198960) | $\Psi_{535}^{(1)}$ | 748 | (1.78236705, 1.78236715) | $\Psi_{748}^{(1)}$ |
| 536 | (1.36817644, 1.274027200) | $\Psi_{536}^{(1)}$ | 750 | (1.78236706, 1.78236716) | $\Psi_{750}^{(1)}$ |
| 537 | (1.36769999, 1.273855440) | $\Psi_{537}^{(1)}$ | 752 | (1.78236707, 1.78236717) | $\Psi_{752}^{(1)}$ |
| 538 | (1.36722354, 1.273683680) | $\Psi_{538}^{(1)}$ | 754 | (1.78236708, 1.78236718) | $\Psi_{754}^{(1)}$ |
| 539 | (1.36674709, 1.273511920) | $\Psi_{539}^{(1)}$ | 756 | (1.78236709, 1.78236719) | $\Psi_{756}^{(1)}$ |
| 540 | (1.36627064, 1.273340160) | $\Psi_{540}^{(1)}$ | 758 | (1.78236710, 1.78236720) | $\Psi_{758}^{(1)}$ |
| 541 | (1.36579419, 1.273168400) | $\Psi_{541}^{(1)}$ | 760 | (1.78236711, 1.78236721) | $\Psi_{760}^{(1)}$ |
| 542 | (1.36531774, 1.272996640) | $\Psi_{542}^{(1)}$ | 762 | (1.78236712, 1.78236722) | $\Psi_{762}^{(1)}$ |
| 543 | (1.36484129, 1.272824880) | $\Psi_{543}^{(1)}$ | 764 | (1.78236713, 1.78236723) | $\Psi_{764}^{(1)}$ |
| 544 | (1.36436484, 1.272653120) | $\Psi_{544}^{(1)}$ | 766 | (1.78236714, 1.78236724) | $\Psi_{766}^{(1)}$ |
| 545 | (1.36388839, 1.272481360) | $\Psi_{545}^{(1)}$ | 768 | (1.78236715, 1.78236725) | $\Psi_{768}^{(1)}$ |
| 546 | (1.36341194, 1.272309600) | $\Psi_{546}^{(1)}$ | 770 | (1.78236716, 1.78236726) | $\Psi_{770}^{(1)}$ |
| 547 | (1.36293549, 1.272137840) | $\Psi_{547}^{(1)}$ | 772 | (1.78236717, 1.78236727) | $\Psi_{772}^{(1)}$ |
| 548 | (1.36245904, 1.271966080) | $\Psi_{548}^{(1)}$ | 774 | (1.78236718, 1.78236728) | $\Psi_{774}^{(1)}$ |
| 549 | (1.36198259, 1.271794320) | $\Psi_{549}^{(1)}$ | 776 | (1.78236719, 1.78236729) | $\Psi_{776}^{(1)}$ |
| 550 | (1.36150614, 1.271622560) | $\Psi_{550}^{(1)}$ | 778 | (1.78236720, 1.78236730) | $\Psi_{778}^{(1)}$ |
| 551 | (1.36102969, 1.271450800) | $\Psi_{551}^{(1)}$ | 780 | (1.78236721, 1.78236731) | $\Psi_{780}^{(1)}$ |
| 552 | (1.36055324, 1.271279040) | $\Psi_{552}^{(1)}$ | 782 | (1.78236722, 1.78236732) | $\Psi_{782}^{(1)}$ |
| 553 | (1.36007679, 1.271107280) | $\Psi_{553}^{(1)}$ | 784 | (1.78236723, 1.78236733) | $\Psi_{784}^{(1)}$ |
| 554 | (1.35960034, 1.270935520) | $\Psi_{554}^{(1)}$ | 786 | (1.78236724, 1.78236734) | $\Psi_{786}^{(1)}$ |
| 555 | (1.35912389, 1.270763760) | $\Psi_{555}^{(1)}$ | 788 | (1.78236725, 1.78236735) | $\Psi_{788}^{(1)}$ |
| 556 | (1.35864744, 1.270592000) | $\Psi_{556}^{(1)}$ | 790 | (1.78236726, 1.78236736) | $\Psi_{790}^{(1)}$ |
| 557 | (1.35817099, 1.270420240) | $\Psi_{557}^{(1)}$ | 792 | (1.78236727, 1.78236737) | $\Psi_{792}^{(1)}$ |
| 558 | (1.35769454, 1.270248480) | $\Psi_{558}^{(1)}$ | 794 | (1.78236728, 1.78236738) | $\Psi_{794}^{(1)}$ |
| 559 | (1.35721809, 1.270076720) | $\Psi_{559}^{(1)}$ | 796 | (1.78236729, 1.78236739) | $\Psi_{796}^{(1)}$ |
| 560 | (1.35674164, 1.269904960) | $\Psi_{560}^{(1)}$ | 798 | (1.78236730, 1.78236740) | $\Psi_{798}^{(1)}$ |
| 561 | (1.35626519, 1.269733200) | $\Psi_{561}^{(1)}$ | 800 | (1.78236731, 1.78236741) | $\Psi_{800}^{(1)}$ |
| 562 | (1.35578874, 1.269561440) | $\Psi_{562}^{(1)}$ | 802 | (1.78236732, 1.78236742) | $\Psi_{802}^{(1)}$ |
| 563 | (1.35531229, 1.269389680) | $\Psi_{563}^{(1)}$ | 804 | (1.78236733, 1.78236743) | $\Psi_{804}^{(1)}$ |
| 564 | (1.35483584, 1.269217920) | $\Psi_{564}^{(1)}$ | 806 | (1.78236734, 1.78236744) | $\Psi_{806}^{(1)}$ |
| 565 | (1.35435939, 1.269046160) | $\Psi_{565}^{(1)}$ | 808 | (1.78236735, 1.78236745) | $\Psi_{808}^{(1)}$ |
| 566 | (1.35388294, 1.268874400) | $\Psi_{566}^{(1)}$ | 810 | (1.78236736, 1.78236746) | $\Psi_{810}^{(1)}$ |
| 567 | (1.35340649, 1.268702640) | $\Psi_{567}^{(1)}$ | 812 | (1.78236737, 1.78236747) | $\Psi_{812}^{(1)}$ |
| 568 | (1.35293004, 1.268530880) | $\Psi_{568}^{(1)}$ | 814 | (1.78236738, 1.78236748) | $\Psi_{814}^{(1)}$ |
| 569 | (1.35245359, 1.268359120) | $\Psi_{569}^{(1)}$ | 816 | (1.78236739, 1.78236749) | $\Psi_{816}^{(1)}$ |
| 570 | (1.35197714, 1.268187360) | $\Psi_{570}^{(1)}$ | 818 | (1.78236740, 1.78236750) | $\Psi_{818}^{(1)}$ |
| 571 | (1.35150069, 1.268015600) | $\Psi_{571}^{(1)}$ | 820 | (1.78236741, 1.78236751) | $\Psi_{820}^{(1)}$ |
| 572 | (1.35102424, 1.267843840) | $\Psi_{572}^{(1)}$ | 822 | (1.78236742, 1.78236752) | $\Psi_{822}^{(1)}$ |
| 573 | (1.35054779, 1.267672080) | $\Psi_{573}^{(1)}$ | 824 | (1.78236743, 1.78236753) | $\Psi_{824}^{(1)}$ |
| 574 | (1.35007134, 1.267500320) | $\Psi_{574}^{(1)}$ | 826 | (1.78236744, 1.78236754) | $\Psi_{826}^{(1)}$ |
| 575 | (1.34959489, 1.267328560) | $\Psi_{575}^{(1)}$ | 828 | (1.78236745, 1.78236755) | $\Psi_{828}^{(1)}$ |
| 576 | (1.34911844, 1.267156800) | $\Psi_{576}^{(1)}$ | 830 | (1.78236746, 1.78236756) | $\Psi_{830}^{(1)}$ |
| 577 | (1.34864199, 1.266985040) | $\Psi_{577}^{(1)}$ | 832 | (1.78236747, 1.78236757) | $\Psi_{832}^{(1)}$ |
| 578 | (1.34816554, 1.266813280) | $\Psi_{578}^{(1)}$ | 834 | (1.78236748, 1.78236758) | $\Psi_{834}^{(1)}$ |
| 579 | (1.34768909, 1.266641520) | $\Psi_{579}^{(1)}$ | 836 | (1.78236749, 1.78236759) | $\Psi_{836}^{(1)}$ |
| 580 | (1.34721264, 1.266469760) | $\Psi_{580}^{(1)}$ | 838 | (1.78236750, 1.78236760) | $\Psi_{838}^{(1)}$ |
| 581 | (1.34673619, 1.266298000) | $\Psi_{581}^{(1)}$ | 840 | (1.78236751, 1.78236761) | $\Psi_{840}^{(1)}$ |
| 582 | (1.34625974, 1.266126240) | $\Psi_{582}^{(1)}$ | 842 | (1.78236752, 1.78236762) | $\Psi_{842}^{(1)}$ |
| 583 | (1.34578329, 1.265954480) | $\Psi_{583}^{(1)}$ | 844 | (1.78236753, 1.78236763) | $\Psi_{844}^{(1)}$ |
| 584 | (1.34530684, 1.265782720) | $\Psi_{584}^{(1)}$ | 846 | (1.78236754, 1.78236764) | $\Psi_{846}^{(1)}$ |
| 585 | (1.34483039, 1.265610960) | $\Psi_{585}^{(1)}$ | 848 | (1.78236755, 1.78236765) | $\Psi_{848}^{(1)}$ |
| 586 | (1.34435394, 1.265439200) | $\Psi_{586}^{(1)}$ | 850 | (1.78236756, 1.78236766) | $\Psi_{850}^{(1)}$ |
| 587 | (1.34387749, 1.265267440) | $\Psi_{587}^{(1)}$ | 852 | (1.78236757, 1.78236767) | $\Psi_{852}^{(1)}$ |
| 588 | (1.34340104, 1.265095680) | $\Psi_{588}^{(1)}$ | 854 | (1.78236758, 1.78236768) | $\Psi_{854}^{(1)}$ |
| 589 | (1.34292459, 1.264923920) | $\Psi_{589}^{(1)}$ | 856 | (1.78236759, 1.78236769) | $\Psi_{856}^{(1)}$ |
| 590 | (1.34244814, 1.264752160) | $\Psi_{590}^{(1)}$ | 858 | (1.78236760, 1.78236770) | $\Psi_{858}^{(1)}$ |
| 591 | (1.34197169, 1.264580400) | $\Psi_{591}^{(1)}$ | 860 | (1.78236761, 1.78236771) | $\Psi_{860}^{(1)}$ |
| 592 | (1.34149524, 1.264408640) | $\Psi_{592}^{(1)}$ | 862 | (1.78236762, 1.78236772) | $\Psi_{862}^{(1)}$ |
| 593 | (1.34101879, 1.264236880) | $\Psi_{593}^{(1)}$ | 864 | (1.78236763, 1.78236773) | $\Psi_{864}^{(1)}$ |
| 594 | (1.34054234, 1.264065120) | $\Psi_{594}^{(1)}$ | 866 | (1.78236764, 1.78236774) | $\Psi_{866}^{(1)}$ |
| 595 | (1.34006589, 1.263893360) | $\Psi_{595}^{(1)}$ | 868 | (1.78236765, 1.78236775) | $\Psi_{868}^{(1)}$ |
| 596 | (1.33958944, 1.263721600) | $\Psi_{596}^{(1)}$ | 870 | (1.78236766, 1.78236776) | $\Psi_{870}^{(1)}$ |
| 597 | (1.33911299, 1.263549840) | $\Psi_{597}^{(1)}$ | 872 | (1.78236767, 1.78236777) | $\Psi_{872}^{(1)}$ |
| 598 | (1.33863654, 1.263378080) | $\Psi_{598}^{(1)}$ | 874 | (1.78236768, 1.78236778) | $\Psi_{874}^{(1)}$ |
| 599 | (1.33816009, 1.263206320) | $\Psi_{599}^{(1)}$ | 876 | (1.78236769, 1.78236779) | $\Psi_{876}^{(1)}$ |
| 600 | (1.33768364, 1.263034560) | $\Psi_{600}^{(1)}$ | 878 | (1.78236770, 1.78236780) | $\Psi_{878}^{(1)}$ |
| 601 | (1.33720719, 1.262862800) | $\Psi_{601}^{(1)}$ | 880 | (1.78236771, 1.78236781) | $\Psi_{880}^{(1)}$ |
| 602 | (1.33673074, 1.262691040) | $\Psi_{602}^{(1)}$ | 882 | (1.78236772, 1.78236782) | $\Psi_{882}^{(1)}$ |
| 603 | (1.33625429, 1.262519280) | $\Psi_{603}^{(1)}$ | 884 | (1.78236773, 1.78236783) | $\Psi_{884}^{(1)}$ |
| 604 | (1.33577784, 1.262347520) | $\Psi_{604}^{(1)}$ | 886 | (1.78236774, 1.78236784) | $\Psi_{886}^{(1)}$ |
| 605 | (1.33530139, 1.262175760) | $\Psi_{605}^{(1)}$ | 888 | (1.78236775, 1.78236785) | $\Psi_{888}^{(1)}$ |
| 606 | (1.33482494, 1.262004000) | $\Psi_{606}^{(1)}$ | 890 | (1.78236776, 1.78236786) | $\Psi_{890}^{(1)}$ |
| 607 | (1.33434849, 1.261832240) | $\Psi_{607}^{(1)}$ | 892 | (1.78236777, 1.78236787) | $\Psi_{892}^{(1)}$ |
| 608 | (1.33387204, 1.261660480) | $\Psi_{608}^{(1)}$ | 894 | (1.78236778, 1.78236788) | $\Psi_{894}^{(1)}$ |
| 609 | (1.33339559, 1.261488720) | $\Psi_{609}^{(1)}$ | 896 | (1.78236779, 1.78236789) | $\Psi_{896}^{(1)}$ |
| 610 | (1.33291914, 1.261316960) | $\Psi_{610}^{(1)}$ | 898 | (1.78236780, 1.78236790) | $\Psi_{898}^{(1)}$ |
| 611 | (1 | | | | |

| Length | Interval of Admissibility | Function | Length | Interval of Admissibility | Function |
|--------|----------------------------|----------------|--------|---------------------------|----------------|
| 82 | [1.27476283, 1.337474294] | Φ_{11}, V | 924 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 83 | [1.27476283, 1.337474294] | Φ_{11}, V | 925 | [1.22945631, 1.22916631] | Φ_{11}, V |
| 84 | [1.27476283, 1.337474294] | Φ_{11}, V | 926 | [1.23148238, 1.23238238] | Φ_{11}, V |
| | [1.273219193, 1.22751167] | Φ_{11}, V | 927 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 85 | [1.273219193, 1.22751167] | Φ_{11}, V | 928 | [1.23148238, 1.23238238] | Φ_{11}, V |
| | [1.242014952, 1.240108973] | Φ_{11}, V | 929 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 86 | [1.242014952, 1.240108973] | Φ_{11}, V | 930 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 87 | [1.242014952, 1.240108973] | Φ_{11}, V | 931 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 88 | [1.242014952, 1.240108973] | Φ_{11}, V | 932 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 89 | [1.242014952, 1.240108973] | Φ_{11}, V | 933 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 90 | [1.242014952, 1.240108973] | Φ_{11}, V | 934 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 91 | [1.242014952, 1.240108973] | Φ_{11}, V | 935 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 92 | [1.242014952, 1.240108973] | Φ_{11}, V | 936 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 93 | [1.242014952, 1.240108973] | Φ_{11}, V | 937 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 94 | [1.242014952, 1.240108973] | Φ_{11}, V | 938 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 95 | [1.242014952, 1.240108973] | Φ_{11}, V | 939 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 96 | [1.242014952, 1.240108973] | Φ_{11}, V | 940 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 97 | [1.242014952, 1.240108973] | Φ_{11}, V | 941 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 98 | [1.242014952, 1.240108973] | Φ_{11}, V | 942 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 99 | [1.242014952, 1.240108973] | Φ_{11}, V | 943 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 100 | [1.242014952, 1.240108973] | Φ_{11}, V | 944 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 101 | [1.242014952, 1.240108973] | Φ_{11}, V | 945 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 102 | [1.242014952, 1.240108973] | Φ_{11}, V | 946 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 103 | [1.242014952, 1.240108973] | Φ_{11}, V | 947 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 104 | [1.242014952, 1.240108973] | Φ_{11}, V | 948 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 105 | [1.242014952, 1.240108973] | Φ_{11}, V | 949 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 106 | [1.242014952, 1.240108973] | Φ_{11}, V | 950 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 107 | [1.242014952, 1.240108973] | Φ_{11}, V | 951 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 108 | [1.242014952, 1.240108973] | Φ_{11}, V | 952 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 109 | [1.242014952, 1.240108973] | Φ_{11}, V | 953 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 110 | [1.242014952, 1.240108973] | Φ_{11}, V | 954 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 111 | [1.242014952, 1.240108973] | Φ_{11}, V | 955 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 112 | [1.242014952, 1.240108973] | Φ_{11}, V | 956 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 113 | [1.242014952, 1.240108973] | Φ_{11}, V | 957 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 114 | [1.242014952, 1.240108973] | Φ_{11}, V | 958 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 115 | [1.242014952, 1.240108973] | Φ_{11}, V | 959 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 116 | [1.242014952, 1.240108973] | Φ_{11}, V | 960 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 117 | [1.242014952, 1.240108973] | Φ_{11}, V | 961 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 118 | [1.242014952, 1.240108973] | Φ_{11}, V | 962 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 119 | [1.242014952, 1.240108973] | Φ_{11}, V | 963 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 120 | [1.242014952, 1.240108973] | Φ_{11}, V | 964 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 121 | [1.242014952, 1.240108973] | Φ_{11}, V | 965 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 122 | [1.242014952, 1.240108973] | Φ_{11}, V | 966 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 123 | [1.242014952, 1.240108973] | Φ_{11}, V | 967 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 124 | [1.242014952, 1.240108973] | Φ_{11}, V | 968 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 125 | [1.242014952, 1.240108973] | Φ_{11}, V | 969 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 126 | [1.242014952, 1.240108973] | Φ_{11}, V | 970 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 127 | [1.242014952, 1.240108973] | Φ_{11}, V | 971 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 128 | [1.242014952, 1.240108973] | Φ_{11}, V | 972 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 129 | [1.242014952, 1.240108973] | Φ_{11}, V | 973 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 130 | [1.242014952, 1.240108973] | Φ_{11}, V | 974 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 131 | [1.242014952, 1.240108973] | Φ_{11}, V | 975 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 132 | [1.242014952, 1.240108973] | Φ_{11}, V | 976 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 133 | [1.242014952, 1.240108973] | Φ_{11}, V | 977 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 134 | [1.242014952, 1.240108973] | Φ_{11}, V | 978 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 135 | [1.242014952, 1.240108973] | Φ_{11}, V | 979 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 136 | [1.242014952, 1.240108973] | Φ_{11}, V | 980 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 137 | [1.242014952, 1.240108973] | Φ_{11}, V | 981 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 138 | [1.242014952, 1.240108973] | Φ_{11}, V | 982 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 139 | [1.242014952, 1.240108973] | Φ_{11}, V | 983 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 140 | [1.242014952, 1.240108973] | Φ_{11}, V | 984 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 141 | [1.242014952, 1.240108973] | Φ_{11}, V | 985 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 142 | [1.242014952, 1.240108973] | Φ_{11}, V | 986 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 143 | [1.242014952, 1.240108973] | Φ_{11}, V | 987 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 144 | [1.242014952, 1.240108973] | Φ_{11}, V | 988 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 145 | [1.242014952, 1.240108973] | Φ_{11}, V | 989 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 146 | [1.242014952, 1.240108973] | Φ_{11}, V | 990 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 147 | [1.242014952, 1.240108973] | Φ_{11}, V | 991 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 148 | [1.242014952, 1.240108973] | Φ_{11}, V | 992 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 149 | [1.242014952, 1.240108973] | Φ_{11}, V | 993 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 150 | [1.242014952, 1.240108973] | Φ_{11}, V | 994 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 151 | [1.242014952, 1.240108973] | Φ_{11}, V | 995 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 152 | [1.242014952, 1.240108973] | Φ_{11}, V | 996 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 153 | [1.242014952, 1.240108973] | Φ_{11}, V | 997 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 154 | [1.242014952, 1.240108973] | Φ_{11}, V | 998 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 155 | [1.242014952, 1.240108973] | Φ_{11}, V | 999 | [1.23148238, 1.23238238] | Φ_{11}, V |
| 156 | [1.242014952, 1.240108973] | Φ_{11}, V | 1000 | [1.23148238, 1.23238238] | Φ_{11}, V |

The table is intended to serve as a benchmark. All the computation has been carried through in rational arithmetic, and all calculations are exact.

Tabulated at the parameter value $a = 0.5$ and for string lengths as far as 200 are :

- the number of strings which may appear as a substring of an admissible sequence
- the number of strings which generate an admissible periodic sequence and the strings themselves

| Length | | | Sequences | | | Length | | | Sequences | | | Length | | | Sequences | | | Length | | | Sequences | | |
|--------|----|------|-----------|------|---|--------|-------|---|-----------|-------|---|--------|-------|---|-----------|-------|---|--------|-------|---|-----------|-------|---|
| 1 | 1 | 3 | 1 | 3 | 1 | 68 | 8563 | 0 | 135 | 41877 | 0 | 68 | 8563 | 0 | 135 | 41877 | 0 | 68 | 8563 | 0 | 135 | 41877 | 0 |
| 2 | 2 | 7 | 1 | 7 | 2 | 70 | 9139 | 0 | 136 | 42597 | 0 | 70 | 9139 | 0 | 136 | 42597 | 0 | 70 | 9139 | 0 | 136 | 42597 | 0 |
| 3 | 3 | 13 | 2 | 13 | 3 | 72 | 9767 | 0 | 137 | 43269 | 0 | 72 | 9767 | 0 | 137 | 43269 | 0 | 72 | 9767 | 0 | 137 | 43269 | 0 |
| 4 | 4 | 20 | 3 | 20 | 4 | 73 | 10393 | 0 | 138 | 43941 | 0 | 73 | 10393 | 0 | 138 | 43941 | 0 | 73 | 10393 | 0 | 138 | 43941 | 0 |
| 5 | 5 | 28 | 4 | 28 | 5 | 74 | 11019 | 0 | 139 | 44613 | 0 | 74 | 11019 | 0 | 139 | 44613 | 0 | 74 | 11019 | 0 | 139 | 44613 | 0 |
| 6 | 6 | 41 | 0 | 41 | 0 | 75 | 11647 | 0 | 140 | 45291 | 0 | 75 | 11647 | 0 | 140 | 45291 | 0 | 75 | 11647 | 0 | 140 | 45291 | 0 |
| 7 | 7 | 55 | 0 | 55 | 0 | 76 | 12281 | 0 | 141 | 45961 | 0 | 76 | 12281 | 0 | 141 | 45961 | 0 | 76 | 12281 | 0 | 141 | 45961 | 0 |
| 8 | 8 | 73 | 0 | 73 | 0 | 77 | 12913 | 0 | 142 | 46633 | 0 | 77 | 12913 | 0 | 142 | 46633 | 0 | 77 | 12913 | 0 | 142 | 46633 | 0 |
| 9 | 9 | 99 | 0 | 99 | 0 | 78 | 13547 | 1 | 143 | 47309 | 0 | 78 | 13547 | 1 | 143 | 47309 | 0 | 78 | 13547 | 1 | 143 | 47309 | 0 |
| 10 | 10 | 125 | 0 | 125 | 0 | 79 | 14181 | 0 | 144 | 47981 | 0 | 79 | 14181 | 0 | 144 | 47981 | 0 | 79 | 14181 | 0 | 144 | 47981 | 0 |
| 11 | 11 | 155 | 0 | 155 | 0 | 80 | 14813 | 0 | 145 | 48653 | 0 | 80 | 14813 | 0 | 145 | 48653 | 0 | 80 | 14813 | 0 | 145 | 48653 | 0 |
| 12 | 12 | 181 | 0 | 181 | 0 | 81 | 15447 | 0 | 146 | 49329 | 0 | 81 | 15447 | 0 | 146 | 49329 | 0 | 81 | 15447 | 0 | 146 | 49329 | 0 |
| 13 | 13 | 213 | 0 | 213 | 0 | 82 | 16081 | 0 | 147 | 50001 | 0 | 82 | 16081 | 0 | 147 | 50001 | 0 | 82 | 16081 | 0 | 147 | 50001 | 0 |
| 14 | 14 | 249 | 0 | 249 | 0 | 83 | 16713 | 0 | 148 | 50673 | 0 | 83 | 16713 | 0 | 148 | 50673 | 0 | 83 | 16713 | 0 | 148 | 50673 | 0 |
| 15 | 15 | 283 | 0 | 283 | 0 | 84 | 17347 | 0 | 149 | 51349 | 0 | 84 | 17347 | 0 | 149 | 51349 | 0 | 84 | 17347 | 0 | 149 | 51349 | 0 |
| 16 | 16 | 317 | 0 | 317 | 0 | 85 | 17981 | 0 | 150 | 52021 | 1 | 85 | 17981 | 0 | 150 | 52021 | 1 | 85 | 17981 | 0 | 150 | 52021 | 1 |
| 17 | 17 | 377 | 0 | 377 | 0 | 86 | 18613 | 0 | 151 | 52693 | 0 | 86 | 18613 | 0 | 151 | 52693 | 0 | 86 | 18613 | 0 | 151 | 52693 | 0 |
| 18 | 18 | 435 | 1 | 435 | 1 | 87 | 19247 | 0 | 152 | 53365 | 0 | 87 | 19247 | 0 | 152 | 53365 | 0 | 87 | 19247 | 0 | 152 | 53365 | 0 |
| 19 | 19 | 493 | 0 | 493 | 0 | 88 | 19881 | 0 | 153 | 54037 | 2 | 88 | 19881 | 0 | 153 | 54037 | 2 | 88 | 19881 | 0 | 153 | 54037 | 2 |
| 20 | 20 | 553 | 0 | 553 | 0 | 89 | 20513 | 0 | 154 | 54709 | 0 | 89 | 20513 | 0 | 154 | 54709 | 0 | 89 | 20513 | 0 | 154 | 54709 | 0 |
| 21 | 21 | 611 | 0 | 611 | 0 | 90 | 21147 | 0 | 155 | 55381 | 0 | 90 | 21147 | 0 | 155 | 55381 | 0 | 90 | 21147 | 0 | 155 | 55381 | 0 |
| 22 | 22 | 673 | 0 | 673 | 0 | 91 | 21781 | 0 | 156 | 56053 | 0 | 91 | 21781 | 0 | 156 | 56053 | 0 | 91 | 21781 | 0 | 156 | 56053 | 0 |
| 23 | 23 | 735 | 0 | 735 | 0 | 92 | 22413 | 0 | 157 | 56725 | 0 | 92 | 22413 | 0 | 157 | 56725 | 0 | 92 | 22413 | 0 | 157 | 56725 | 0 |
| 24 | 24 | 800 | 0 | 800 | 0 | 93 | 23047 | 0 | 158 | 57397 | 0 | 93 | 23047 | 0 | 158 | 57397 | 0 | 93 | 23047 | 0 | 158 | 57397 | 0 |
| 25 | 25 | 881 | 0 | 881 | 0 | 94 | 23681 | 0 | 159 | 58069 | 0 | 94 | 23681 | 0 | 159 | 58069 | 0 | 94 | 23681 | 0 | 159 | 58069 | 0 |
| 26 | 26 | 957 | 0 | 957 | 0 | 95 | 24313 | 0 | 160 | 58741 | 0 | 95 | 24313 | 0 | 160 | 58741 | 0 | 95 | 24313 | 0 | 160 | 58741 | 0 |
| 27 | 27 | 1037 | 2 | 1037 | 2 | 96 | 24947 | 0 | 161 | 59413 | 0 | 96 | 24947 | 0 | 161 | 59413 | 0 | 96 | 24947 | 0 | 161 | 59413 | 0 |
| 28 | 28 | 1125 | 0 | 1125 | 0 | 97 | 25581 | 0 | 162 | 60085 | 0 | 97 | 25581 | 0 | 162 | 60085 | 0 | 97 | 25581 | 0 | 162 | 60085 | 0 |
| 29 | 29 | 1213 | 0 | 1213 | 0 | 98 | 26213 | 0 | 163 | 60757 | 0 | 98 | 26213 | 0 | 163 | 60757 | 0 | 98 | 26213 | 0 | 163 | 60757 | 0 |
| 30 | 30 | 1300 | 0 | 1300 | 0 | 99 | 26847 | 0 | 164 | 61429 | 0 | 99 | 26847 | 0 | 164 | 61429 | 0 | 99 | 26847 | 0 | 164 | 61429 | 0 |
| 31 | 31 | 1409 | 0 | 1409 | 0 | 100 | 27481 | 0 | 165 | 62101 | 0 | 100 | 27481 | 0 | 165 | 62101 | 0 | 100 | 27481 | 0 | 165 | 62101 | 0 |
| 32 | 32 | 1509 | 0 | 1509 | 0 | 101 | 28113 | 0 | 166 | 62773 | 0 | 101 | 28113 | 0 | 166 | 62773 | 0 | 101 | 28113 | 0 | 166 | 62773 | 0 |
| 33 | 33 | 1609 | 0 | 1609 | 0 | 102 | 28747 | 0 | 167 | 63445 | 0 | 102 | 28747 | 0 | 167 | 63445 | 0 | 102 | 28747 | 0 | 167 | 63445 | 0 |
| 34 | 34 | 1713 | 0 | 1713 | 0 | 103 | 29381 | 0 | 168 | 64117 | 0 | 103 | 29381 | 0 | 168 | 64117 | 0 | 103 | 29381 | 0 | 168 | 64117 | 0 |
| 35 | 35 | 1824 | 0 | 1824 | 0 | 104 | 30013 | 0 | 169 | 64789 | 0 | 104 | 30013 | 0 | 169 | 64789 | 0 | 104 | 30013 | 0 | 169 | 64789 | 0 |
| 36 | 36 | 1946 | 0 | 1946 | 0 | 105 | 30647 | 0 | 170 | 65461 | 0 | 105 | 30647 | 0 | 170 | 65461 | 0 | 105 | 30647 | 0 | 170 | 65461 | 0 |
| 37 | 37 | 2081 | 0 | 2081 | 0 | 106 | 31281 | 0 | 171 | 66133 | 0 | 106 | 31281 | 0 | 171 | 66133 | 0 | 106 | 31281 | 0 | 171 | 66133 | 0 |
| 38 | 38 | 2217 | 1 | 2217 | 1 | 107 | 31913 | 0 | 172 | 66805 | 0 | 107 | 31913 | 0 | 172 | 66805 | 0 | 107 | 31913 | 0 | 172 | 66805 | 0 |
| 39 | 39 | 2357 | 0 | 2357 | 0 | 108 | 32547 | 0 | 173 | 67477 | 0 | 108 | 32547 | 0 | 173 | 67477 | 0 | 108 | 32547 | 0 | 173 | 67477 | 0 |
| 40 | 40 | 2503 | 0 | 2503 | 0 | 109 | 33181 | 0 | 174 | 68149 | 0 | 109 | 33181 | 0 | 174 | 68149 | 0 | 109 | 33181 | 0 | 174 | 68149 | 0 |
| 41 | 41 | 2653 | 0 | 2653 | 0 | 110 | 33813 | 0 | 175 | 68821 | 0 | 110 | 33813 | 0 | 175 | 68821 | 0 | 110 | 33813 | 0 | 175 | 68821 | 0 |
| 42 | 42 | 2809 | 2 | 2809 | 2 | 111 | 34447 | 0 | 176 | 69493 | 0 | 111 | 34447 | 0 | 176 | 69493 | 0 | 111 | 34447 | 0 | 176 | 69493 | 0 |
| 43 | 43 | 2969 | 0 | 2969 | 0 | 112 | 35081 | 0 | 177 | 70165 | 0 | 112 | 35081 | 0 | 177 | 70165 | 0 | 112 | 35081 | 0 | 177 | 70165 | 0 |
| 44 | 44 | 3138 | 0 | 3138 | 0 | 113 | 35713 | 0 | 178 | 70837 | 0 | 113 | 35713 | 0 | 178 | 70837 | 0 | 113 | 35713 | 0 | 178 | 70837 | 0 |
| 45 | 45 | 3305 | 0 | 3305 | 0 | 114 | 36347 | 0 | 179 | 71509 | 0 | 114 | 36347 | 0 | 179 | 71509 | 0 | 114 | 36347 | 0 | 179 | 71509 | 0 |
| 46 | 46 | 3479 | 0 | 3479 | 0 | 115 | 36981 | 0 | 180 | 72181 | 0 | 115 | 36981 | 0 | 180 | 72181 | 0 | 115 | 36981 | 0 | 180 | 72181 | 0 |
| 47 | 47 | 3668 | 0 | 3668 | 0 | 116 | 37613 | 0 | 181 | 72853 | 0 | 116 | 37613 | 0 | 181 | 72853 | 0 | 116 | 37613 | 0 | 181 | 72853 | 0 |
| 48 | 48 | 3861 | 0 | 3861 | 0 | 117 | 38247 | 0 | 182 | 73525 | 0 | 117 | 38247 | 0 | 182 | 73525 | 0 | 117 | 38247 | 0 | 182 | 73525 | 0 |
| 49 | 49 | 4073 | 0 | 4073 | 0 | 118 | 38881 | 0 | 183 | 74197 | 0 | 118 | 38881 | 0 | 183 | 74197 | 0 | 118 | 38881 | 0 | 183 | 74197 | 0 |
| 50 | 50 | 4298 | 0 | 4298 | 0 | 119 | 39513 | 0 | 184 | 74869 | 0 | 119 | 39513 | 0 | 184 | 74869 | 0 | 119 | 39513 | 0 | 184 | 74869 | 0 |
| 51 | 51 | 4425 | 0 | 4425 | 0 | 120 | 40147 | 0 | 185 | 75541 | 0 | 120 | 40147 | 0 | 185 | 75541 | 0 | 120 | 40147 | 0 | 185 | 75541 | 0 |
| 52 | 52 | 4623 | 0 | 4623 | 0 | 121 | 40781 | 0 | 186 | 76213 | 0 | 121 | 40781 | 0 | 186 | 76213 | 0 | 121 | 40781 | 0 | 186 | 76213 | 0 |
| 53 | 53 | 4833 | 0 | 4833 | 0 | 122 | 41413 | 0 | 187 | 76885 | 0 | 122 | 41413 | 0 | 187 | 76885 | 0 | 122 | 41413 | 0 | 187 | 76885 | 0 |
| 54 | 54 | 5059 | 0 | 5059 | 0 | 123 | 42047 | 0 | 188 | 77557 | 0 | 123 | 42047 | 0 | 188 | 77557 | 0 | 123 | 42047 | 0 | 188 | 77557 | 0 |
| 55 | 55 | 5290 | 0 | 5290 | 0 | 124 | 42681 | 0 | 189 | 78229 | 0 | 124 | 42681 | 0 | 189 | 78229 | 0 | 124 | 42681 | 0 | 189 | 78229 | 0 |
| 56 | 56 | 5519 | 0 | 5519 | 0 | 125 | 43313 | 0 | 190 | 78901 | 0 | 125 | 43313 | 0 | 190 | 78901 | 0 | 125 | 43313 | 0 | 190 | 78901 | 0 |
| 57 | 57 | 5749 | 0 | 5749 | 0 | 126 | 43947 | 0 | 191 | 79573 | 0 | 126 | 43947 | 0 | 191 | 79573 | 0 | 126 | 43947 | 0 | 191 | 79573 | 0 |
| 58 | 58 | 5983 | 0 | 5983 | 0 | 127 | 44581 | 0 | 192 | 80245 | 0 | 127 | 44581 | 0 | 192 | 80245 | 0 | 127 | 44581 | 0 | 192 | 80245 | 0 |
| 59 | 59 | 6231 | 0 | 6231 | 0 | 128 | 45213 | 0 | 193 | 80917 | 0 | 128 | 45213 | 0 | 193 | 80917 | 0 | 128 | 45213 | 0 | 193 | 80917 | 0 |
| 60 | 60 | 6485 | 0 | 6485 | 0 | 129 | 45847 | 0 | 194 | 81589 | 0 | 129 | 45847 | 0 | 194 | 81589 | 0 | 129 | 45847 | 0 | 194 | 81589 | 0 |
| 61 | 61 | 6745 | 0 | 6745 | 0 | 130 | 46481 | 0 | 195 | 82261 | 0 | 130 | 46481 | 0 | 195 | 82261 | 0 | 130 | 46481 | 0 | 195 | 82261 | 0 |
| 62 | 62 | 6999 | 0 | 6999 | 0 | 131 | 47113 | 0 | 196 | 82933 | 0 | 131 | 47113 | 0 | 196 | 82933 | 0 | 131 | 47113 | 0 | 196 | 82933 | 0 |
| 63 | 63 | 7263 | 0 | 7263 | 0 | 132 | 47747 | 0 | 197 | 83605 | 0 | 132 | 47747 | 0 | 197 | 83605 | 0 | 132 | 47747 | 0 | 197 | 83605 | 0 |
| 64 | 64 | 7533 | 0 | 7533 | 0 | 133 | 48381 | 0 | 198 | 84277 | 0 | 133 | 48381 | 0 | 198 | 84277 | 0 | 133 | 48381 | 0 | 198 | 84277 | 0 |
| 65 | 65 | 7809 | 2 | 7809 | 2 | 134 | 49013 | 0 | 199 | 84949 | 0 | 134 | 49013 | 0 | 199 | 84949 | 0 | 134 | 49013 | 0 | 199 | 84949 | 0 |
| 66 | 66 | | | | | | | | | | | | | | | | | | | | | | |

Admissible Periodic Sequences : $a = 0.5$

Listed up to length 3000

```

1 | 0
2 | 0
3 | 0
4 | 0
5 | 0
6 | 0
7 | 0
8 | 0
9 | 0
10 | 0
11 | 0
12 | 0
13 | 0
14 | 0
15 | 0
16 | 0
17 | 0
18 | 0
19 | 0
20 | 0
21 | 0
22 | 0
23 | 0
24 | 0
25 | 0
26 | 0
27 | 0
28 | 0
29 | 0
30 | 0
31 | 0
32 | 0
33 | 0
34 | 0
35 | 0
36 | 0
37 | 0
38 | 0
39 | 0
40 | 0
41 | 0
42 | 0
43 | 0
44 | 0
45 | 0
46 | 0
47 | 0
48 | 0
49 | 0
50 | 0
51 | 0
52 | 0
53 | 0
54 | 0
55 | 0
56 | 0
57 | 0
58 | 0
59 | 0
60 | 0
61 | 0
62 | 0
63 | 0
64 | 0
65 | 0
66 | 0
67 | 0
68 | 0
69 | 0
70 | 0
71 | 0
72 | 0
73 | 0
74 | 0
75 | 0
76 | 0
77 | 0
78 | 0
79 | 0
80 | 0
81 | 0
82 | 0
83 | 0
84 | 0
85 | 0
86 | 0
87 | 0
88 | 0
89 | 0
90 | 0
91 | 0
92 | 0
93 | 0
94 | 0
95 | 0
96 | 0
97 | 0
98 | 0
99 | 0

```

[illegible]

Appendix A Further Investigations

90

intervals, obtained from the congruences (3.1) and (3.49), where these strings may appear within an admissible sequence, do not include $(\frac{\pi}{2} - \epsilon, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{\pi}{2} + \epsilon)$ for any $\epsilon > 0$. We are concerned then only with a periodic sequence admissible in an interval, one of whose end-points is $\frac{\pi}{2}$.

In Chapter 5 we established that the following families of strings have an interval of admissibility with one end-point adjacent to $\frac{\pi}{2}$.

| | | |
|-------------------|-----------------------------------|--|
| +0...0 and -0...0 | of lengths $N = 1, 5, 9, \dots$ | with interval $(\frac{\pi}{2}, \frac{(N+1)\pi}{2N})$ |
| +0...0 and -0...0 | of lengths $N = 3, 7, 11, \dots$ | with interval $(\frac{(N-1)\pi}{2N}, \frac{\pi}{2})$ |
| +0...0-0...0 | of lengths $N = 6, 14, 22, \dots$ | with interval $(\frac{\pi}{2}, \frac{(N+2)\pi}{2N})$ |
| +0...0-0...0 | of lengths $N = 2, 10, 18, \dots$ | with interval $(\frac{(N-2)\pi}{2N}, \frac{\pi}{2})$ |

If we refer to the results of our computer searches for admissible periodic sequences, as reported in Table 1 of Volume II, the lists corresponding to the values $\theta = 1.55$ and $\theta = 1.6$, close to and either side of $\frac{\pi}{2}$, suggest that these families account entirely for periodic sequences admissible adjacent to $\frac{\pi}{2}$. We find then that the task here is of a different nature to the classification dealt with in Chapters 7 and 8; there we needed to produce all the strings admissible adjacent to zero, whereas now we aim to show that the list we already possess is complete.

The starting point is again the Chua-Lin inequalities. It turns out that the restrictions they impose at $\frac{\pi}{2}$ are of quite a different character to those at zero, and different methods are needed in the new situation.

A.1.1 The differential form of the Chua-Lin inequalities

To keep the exposition simple, we suppose for the moment $N = 9$; the general pattern is easily inferred from this particular case. Then

$$g(\theta) = \sin^2 \theta \sin \theta,$$

Appendix A

Further Investigations

This appendix outlines other investigations conducted during the period of research for this thesis. Due to limitations of space, it is not possible to present them in full. The purpose of their inclusion here is primarily to establish the priority of the author over the work described. The emphasis is on reporting the main results, the key steps in the arguments and crucial formulae, but routine calculations have been suppressed in the interests of brevity.

A.1 Periodic sequences admissible adjacent to $\frac{\pi}{2}$

We attempt here a similar task to that undertaken in Chapters 7 and 8, but at $\frac{\pi}{2}$ instead of at zero. That is, we wish to determine the periodic sequences having an interval of admissibility extending to $\frac{\pi}{2}$.

A straightforward consequence of the results in Chapter 3 about strings which appear in admissible sequences is that, apart from the trivial sequence entirely composed of zero digits, no periodic sequence has an interval of admissibility that straddles $\frac{\pi}{2}$. For if there could exist such a sequence, it would have all nonzero digits isolated, that is, separated by a run of one or more zeros, because any pair of nonzero digits (i.e. +-, -+, ++ and --) is forbidden from appearing in an admissible sequence for one of the half-ranges $0 < \theta < \frac{\pi}{2}$ or $\frac{\pi}{2} < \theta < \pi$. So the sequence would contain a substring consisting of two nonzero digits separated by a run of zeros (i.e. one of +0...0-, -0...0+, +0...0+, -0...0-). But the

-1 1 1 1 -1 $\Delta \dots$ 1
 \uparrow
 -1 1 1 1 -1 $\Delta \dots$ 1 or $N=6$ and the strings is -1 1 1 1 -1 1 1 i.e. +00+00
 or -1 1 1 1 -1 $\Delta \dots$ 1

The right-hand branch gives a repeated +00 pattern for $N=6, 18, 30, \dots$. The left-hand branch continues:

-1 1 1 1 -1 1 $\Delta \dots$ 1
 \uparrow
 -1 1 1 1 -1 1 1 $\Delta \dots$ 1 or -1 1 1 1 -1 1 1 -1 $\Delta \dots$ 1

In this instance, the right-hand branch when developed gives the strings consisting of repetitions of +0000-0000 for $N=10, 30, 50, \dots$.

In this way we recover the strings generating periodic sequences admissible in an interval adjacent to $\frac{1}{2}$ on the left that were described in Chapter 5, either as multiple copies of the irreducible strings +0...0 of lengths 3, 7, 11, ..., or as multiple copies of the strings +0...0-0...0 obtained from the irreducible strings +0...0 of lengths 5, 9, 13, ... by doubling and negation of alternate digits.

The second possibility involving run lengths 2 and 3, is analysed in exactly similar fashion, or reduced to the previous case by the process of alternate digit negation, and recovers the strings known to generate periodic sequences admissible in an interval adjacent to $\frac{1}{2}$ on the right.

A.1.5 The case when N is twice an even number

The situation when N is twice an even number, say $N=4m$, is considerably more complex than the cases previously considered. Indeed, that the strings of length N generating periodic sequences admissible adjacent to $\frac{1}{2}$ are no more than the multiples of those irreducible strings identified in Chapter 5 is not yet shown. We present briefly the results that we have been able to obtain.

From $g(\theta) = \sin \frac{N}{2}\theta \sin \theta$, we find $g = g(\frac{\pi}{2}) = 0$ and $g' = g'(\frac{\pi}{2}) = (-1)^m \frac{N}{2}$. It then

follows from the Chuva-Lin inequalities that each $f_r = f_r(\frac{\pi}{2}) = 0$, and that $-\frac{N}{2} \leq f'_r = f'_r(\frac{\pi}{2}) \leq \frac{N}{2}$. We have the relations $f'_{r-1} + f'_{r+1} = (-1)^m N \delta_r$ and $f''_{r-1} + f''_{r+1} = -4f'_r$. An argument modelled on that used in Section A.1.4, involving eigenvalues of a symmetric circulant matrix (for showing there that each $f_r = \pm 1$), can be adapted to show that now each $f'_r = \pm \frac{N}{2}$.

A string $s_0 s_1 \dots s_{N-1}$, of length $N=4m$, that generates a periodic sequence admissible adjacent to $\frac{1}{2}$ can be described in terms of the string of derivative values $f'_0 f'_1 \dots f'_{N-1}$. A natural step is to attempt to establish results analogous to those for the case when N is twice an odd number, and we find that the substrings $\frac{N}{2} \frac{N}{2} \frac{N}{2}$ and $\frac{N}{2} -\frac{N}{2} \frac{N}{2}$ cannot coexist as substrings of a valid generating string (likewise for $\frac{N}{2} \frac{N}{2} \frac{N}{2}$ and $-\frac{N}{2} \frac{N}{2} \frac{N}{2}$). We have been unable to exclude $\frac{N}{2} \frac{N}{2} \frac{N}{2} \frac{N}{2}$ as a substring of a valid generating string.

A.2 Dynamics of a one-dimensional reduction of the filter map

Our second topic explores some aspects of the behaviour of the map modelling a digital filter with 2's complement overflow, but now when its parameters are situated in a different region of the (a, b) parameter space. With the terminology introduced in Section 2.3.1, the filter map is $\mathbf{F}_{a,b}$ and the region under investigation is R_5 ; here the map expands in one direction and contracts in a second. Our study is specialised to the particular case when $b=0$, because of the definition of R_5 this restricts a so that $|a| > 1$. We shall find that for such parameter values the dynamics of $\mathbf{F}_{a,b}$ can be investigated via a one-dimensional map.

When $b=0$, the linear map associated with $\mathbf{F}_{a,b}$ is $\mathbf{x} \mapsto \begin{pmatrix} 0 & 1 \\ a & 0 \end{pmatrix} \mathbf{x}$, whose matrix has eigenvalues $a, 0$ and corresponding eigenvectors $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. An arbitrary point $\mathbf{x} = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2$ in state-space is taken by the linear map to $a\lambda_1 \mathbf{e}_1$, and consequently one application of the map $\mathbf{F}_{a,b}$ sends all points in I^2 onto the set

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1 \in [-1, 1], x_2 = (ax_1 + 1) \bmod 2 - 1 \right\}. \quad (\text{A.13})$$

a finite collection of parallel line segments in I^2 , each of slope a . Thus the entire orbit of any point under $\mathbf{F}_{a,b}$, apart from the initial point, lies in S . Furthermore, since the projection of S onto the x_1 coordinate axis is a bijection, we can study $\mathbf{F}_{a,b}$ in terms of the map

$$f(x) = (ax + 1) \bmod 2 - 1. \quad (\text{A.14})$$

Graphs of f are given in Figure A.1; note that it is piecewise linear with constant slope except for the discontinuities which occur at $x = \pm \frac{2i-1}{a}$ for $i = 1, 2, \dots, \lfloor \frac{1}{2}(a+1) \rfloor$.

There is a technical difficulty arising from the half-open character of the range of x , and from the points of discontinuity of f . This may be overcome by excising from the interval $[-1, 1]$ the countable set of all the images and pre-images of the discontinuities under the compositions f^n . Nevertheless, in describing subsets of the resulting domain we will need subintervals of $[-1, 1]$ whose end-points lie within this excised set. Correspondingly whenever we subsequently refer to an interval we intend the intersection of that interval with the domain, i.e. the trace of the interval on the domain. The function f is an odd function when restricted to this domain.

In the work that follows, we investigate the topological dynamics of the map f , and our focus is particularly on the issue of topological transitivity. We take $a > 1$ as a standing assumption; the dynamics of the map when a is negative is easily inferred.

The situation is relatively straightforward when $a > 2$; in this case the map is topologically mixing on the whole interval $[-1, 1]$. Our method of proof is similar to that used to prove that the discontinuous Lorenz maps are topologically transitive (see, for example, Guckenheimer & Holmes (1983)); we show that any small interval $J \subset [-1, 1]$ expands to cover $[-1, 1]$ after a finite number of iterations under f . Dependent on the size and location of J there are three possible cases to consider.

- (i) J includes none of the discontinuities, so is not split when mapped by f . Thus $f(J)$ has just one component whose length is $|f(J)| = a|J| = 2|J|$.
- (ii) $|J| \geq \frac{2}{a}$; this implies $|f(J)| = [-1, 1]$.

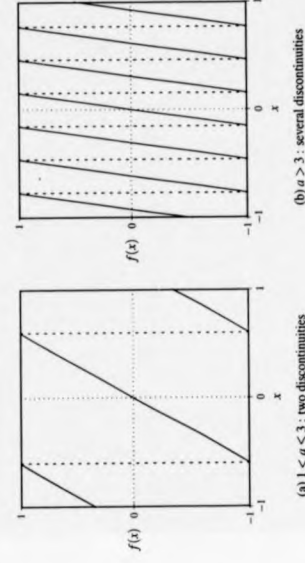


Figure A.1: Graphs of the one-dimensional map f .

- (iii) $|J| < \frac{2}{a}$ and J includes a single discontinuity, so the image $f(J)$ consists of two intervals. Let J_{\max} be the larger of the two components of J that lie to either side of the discontinuity. Then $|f(J_{\max})| \geq \frac{1}{2}a|J| > |J|$.

In each case, one component of the image $f(J)$ is an interval whose length exceeds that of J , so after a finite number of iterations the image will expand to cover $[-1, 1]$.

For parameter values in the range $1 < a \leq 2$, the issue of topological transitivity for f becomes very much more complicated. The argument above is no longer valid: the problem arises in case (iii) where $\frac{2}{a} \leq 1$ and so the interval $f(J_{\max})$ need not be longer than J . Computer simulation reveals that the behaviour of f becomes increasingly complex as a approaches 1: Figure A.2 exhibits plots of orbits for a selection of values of a . Indeed we are able to show that the map undergoes a sequence of bifurcations at the values a where $a^{2^n} = 2$, for $n = 0, 1, 2, \dots$, and that f has an attracting set whose structure undergoes abrupt changes at these values, becoming increasingly complex as a approaches 1. A key tool in the work is the identification of a renormalisation for f ; for positive integers i and j , the functions f^{2^i} and f^{2^j} , taken over the respective parameter ranges $\sqrt{2} < a^{2^i} < 2$ and

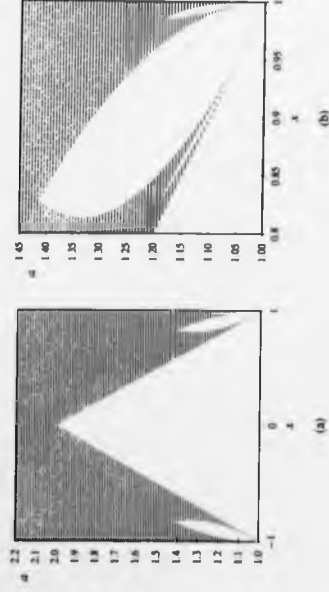


Figure A.2: The change in the nature of the orbits of f as the parameter a is varied. Each horizontal section across the plots shows the effect of iterating the point $x = 0.3$; in each case the first 1000 iterates have been suppressed so that the plotted points reveal the structure of the intervals comprising the attracting set. As a decreases through the values 2, 2.1, 2.4, ..., each interval splits to form two new intervals; in plot (a) the first two splittings can be seen, and the increased resolution of plot (b), drawn over part of the range $[-1, 1]$, shows the third splitting.

$\sqrt{2} < a^2 < 2$, share some important features, and this allows us to develop arguments that can be re-applied within each of the successive intervals of the parameter range.

A.2.1 An invariant set on which f is topologically transitive

When the parameter a is decreased through $a = 2$, the map f fails to be topologically transitive on the whole interval $[-1, 1]$. For example, when $1 < a < 2$ we may define intervals $U = [2 - a, 1]$ and $V = [-\epsilon, 1]$ for any $\epsilon < 2 - a$, so that $f^n(U)$ and V never intersect. We show in this section, that for parameter values in the range $1 < a < 2$ it is appropriate instead to consider the restriction of f to some subset of $[-1, 1]$ consisting of a finite union of intervals, on this subset f is topologically transitive.

Proposition A.8

$f^2(I_0^+ \cup I_0^-) \subset I_0^+ \cup I_0^-$ whenever $a^2 \leq 2$.

We obtain through $I_0^+ \cup I_0^-$ a suitable domain on which to study the behaviour of f . Our first major result in this section is the following proposition.

Proposition A.9

f^2 is topologically mixing on the invariant set $I_0^+ \cup I_0^-$ for $\sqrt{2} < a^2 \leq 2$.

The crucial step in the proof of Proposition A.9 is the following lemma, showing that after three applications of f^2 , the image of the interval $[d_n, 1]$ covers the whole set $I_0^+ \cup I_0^-$.

Lemma A.10

If $\sqrt{2} < a^2 \leq 2$ then $(f^2)^{(3)}([d_n, 1]) = (f^2)^{(3)}([-1, -d_n]) = I_0^+ \cup I_0^-$.

Proof

Figure A.4 shows the effect of three successive applications of f^2 on the interval $[d_n, 1]$. From the final diagram, corresponding to the third function application, it is clear that $(f^2)^{(3)}([d_n, 1]) \supset I_0^-$. To show also that $(f^2)^{(3)}([d_n, 1]) \supset I_0^+$, it remains to demonstrate that $f^2(f^2(-k)) \geq f^2(k)$, where the quantity $k = f^2(-1) = 1 - a^2(1 - d_n)$ is the location of the left end-point of I_0^- . Writing $t = f^2(k)$, and using the relationships

$$f^2(t) = -k - a^2(1 - t) \quad (\text{A.17a})$$

$$t = f^2(k) = 1 - (a^2 - 1)(1 - k), \quad (\text{A.17b})$$

derived from the graph of f^2 shown in Figure A.3, the condition $f^2(f^2(-k)) \geq f^2(k)$ becomes

$$a^2(a^2 - 1)(1 - k) + k \geq 1 - (a^2 - 1)(1 - k), \quad (\text{A.18})$$

and this in turn simplifies to $(a^2)^2 \geq 2$.

The proof of Proposition A.9 is completed by showing that any subinterval $J \subset I_0^+ \cup I_0^-$

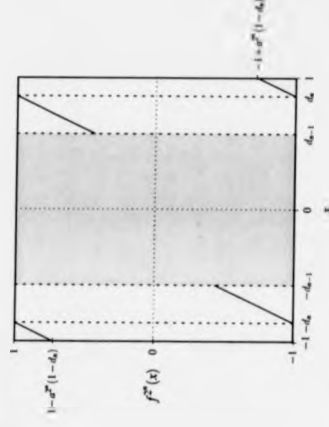


Figure A.3: The form of the function f^2 in the region $|x| > d_{n-1}$, when $a^{2^{n-1}} < \frac{1}{2}\sqrt{2}$.

Proposition A.7

For $a^{2^{n-1}} \leq \frac{1}{2}\sqrt{2}$, the function f^2 , which is piecewise linear with constant slope a^{2^n} , has precisely two discontinuities within the domain $|x| > d_{n-1}$, and they are located at $x = \pm d_n$, where the quantities d_{n-1} and d_n are defined via the recurrence

$$d_0 = \frac{1}{a}, \quad d_n = d_{n-1} + \frac{1 - d_{n-1}}{a^{2^n}}. \quad (\text{A.15})$$

The form of f^2 in the domain $|x| > d_{n-1}$ is as shown in Figure A.3.

When $a^{2^{n-1}} \leq \frac{1}{2}\sqrt{2}$, we can define a pair of intervals I_0^+, I_0^- in the region $|x| > d_{n-1}$ by

$$I_0^+ = [1 - a^{2^n}(1 - d_n), 1], \quad I_0^- = [-1, -1 + a^{2^n}(1 - d_n)]. \quad (\text{A.16})$$

If a satisfies the more restrictive condition $a^{2^n} \leq 2$, then the disjoint union of these two intervals, $I_0^+ \cup I_0^-$, is invariant under f^2 .

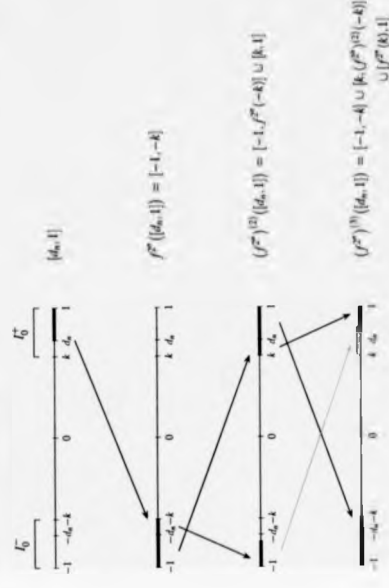


Figure A.4: A diagrammatic representation of the effect of f^2 on the interval $[d_n, 1]$, for the proof of Lemma A.10.

grows to cover $I_0^+ \cup I_0^-$ after a finite number of iterations of the map f^2 . Either J does not straddle one of the discontinuities $\pm d_n$, in which case $|f^2(J)| > \sqrt{2}|J|$, or else we let J_{\max} be the longer of the two components of J lying to either side of the discontinuity: $f^2(J_{\max})$ is one of the intervals

$$[-1, -d_n] \quad \text{or} \quad [-1, 0] \quad \text{or} \quad [0, d_n] \quad \text{or} \quad [d_n, 1]$$

dependent on which discontinuity is involved, $-d_n$ or d_n , and on which side of the discontinuity J_{\max} is situated. If the end-point c of $f^2(J_{\max})$ reaches beyond the discontinuity, then the lemma implies that after three further iterations, $f^2(J_{\max})$ covers $I_0^+ \cup I_0^-$. The other possibility is that the interval $f^2(J_{\max})$ does not extend as far as the discontinuity (as drawn in the sketch of the intervals above), and thus is not split when f^2 is re-applied.

Then

$$|(f^{2^n})^{(2)}(J_{\max})| = \sigma^{2^{n+1}} J_{\max} \geq \frac{1}{2} \sigma^{2^{n+1}} |J| > |J| \quad (\text{A.19})$$

since $\sigma^{2^n} > \sqrt{2}$.

The disjoint union $I_0^+ \cup I_0^-$ is not the only pair of intervals on which f^{2^n} is topologically mixing. We define $2^n - 1$ additional pairs of sets as:

$$I_1^+ = f(I_0^+), \quad I_1^- = f^2(I_0^+), \quad \dots, \quad I_{2^n-1}^+ = f^{2^n-1}(I_0^+) \quad (\text{A.20a})$$

$$I_1^- = f(I_0^-), \quad I_2^- = f^2(I_0^-), \quad \dots, \quad I_{2^n-1}^- = f^{2^n-1}(I_0^-). \quad (\text{A.20b})$$

Proposition A.11

Each of the sets $I_0^+, I_1^+, \dots, I_{2^n-1}^+, I_0^-, I_1^-, \dots, I_{2^n-1}^-$ is a single interval in $[-1, 1]$.

To establish this result we must show that I_0^+ and I_0^- are not split by f^i for each $i \leq 2^n - 1$. An apparent difficulty here is that we are concerned with functions f^i when i is not a power of 2, and we know nothing about the form of such functions. The difficulty is resolved by writing f^i as the composition of functions f^{2^j} ; letting $b_{n-1}b_{n-2}\dots b_1b_0$ denote the n digit binary representation of i , the set I_i^+ is given by

$$I_i^+ = (\phi_0 \circ \phi_1 \circ \dots \circ \phi_{n-1})(I_0^+) \quad \text{where } \phi_i = \begin{cases} f^{2^i} & \text{if } b_i = 1, \\ \text{id} & \text{if } b_i = 0 \end{cases} \quad (\text{identity function}).$$

The interval I_0^+ is in a part of the domain where the form of ϕ_{n-1} is known; likewise $\phi_{n-1}(I_0^+)$ is in a part of the domain where the form of ϕ_{n-2} is known, and so on down to ϕ_0 . At each stage we can show that the extra function application does not cause the interval to be split. Using the fact that f is an odd function, the result transfers immediately to the sets I_i^- .

We are now able to extend Proposition A.9 to these additional pairs of intervals. The new result follows because any subinterval of I_i^+ or I_i^- originates via f^i from a subinterval

of I_0^+ or I_0^- , and from the knowledge that this latter subinterval expands under iteration by f^{2^n} to cover the whole of $I_0^+ \cup I_0^-$.

Proposition A.12

For $\sqrt{2} < \sigma^{2^n} \leq 2$, the map f^{2^n} is topologically mixing on the interval pair $I_i^+ \cup I_i^-$, where $i = 0, 1, \dots, 2^n - 1$.

Given that none of the intervals are split when mapped by f , it follows immediately that $|I_i^+| = |I_i^-| = \sigma^i |I_0^+|$, so that in the sequence of intervals $I_0^+, I_1^+, \dots, I_{2^n-1}^+$ each interval is longer than its predecessor by the factor σ . This information, together with the topological mixing property, can be used to show that the intervals do not overlap.

Proposition A.13

The intervals $I_0^+, I_1^+, \dots, I_{2^n-1}^+, I_0^-, I_1^-, \dots, I_{2^n-1}^-$ are disjoint, or else only share end-points.

We summarise our results on the topological transitivity of f in the following proposition.

Proposition A.14

For $\sqrt{2} < \sigma^{2^n} \leq 2$, f is topologically transitive on an invariant set $\Lambda_n = \bigcup_{i=0}^{2^n-1} I_i^+ \cup I_i^-$, a collection of 2^{n+1} intervals that are either pairwise disjoint or only share end-points. The map f^{2^n} is topologically mixing on each interval pair $I_i^+ \cup I_i^-$ and f permutes the pairs of intervals.

A.2.2 Dynamics of points outside of Λ_n

The aim in this section is to show that when the parameter a is in the range specified by $\sqrt{2} < \sigma^{2^n} \leq 2$, the set Λ_n , as defined in Proposition A.14 above, is the only part of the interval $[-1, 1]$ on which the dynamics of f is interesting: almost all points in the remainder $[-1, 1] \setminus \Lambda_n$ enter Λ_n after a finite number of iterations.

In the previous section we saw that as a is reduced through $\sigma^{2^n} = 2$, f ceases to be topologically transitive on the disjoint union of 2^n intervals Λ_{n-1} , and that this prompts

the introduction of a new set Λ_n consisting of the disjoint union of 2^{n+1} intervals. It is the case, however, that the intervals making up Λ_{n-1} remain defined for all $\sigma^{2^n} < 2$, and the set Λ_{n-1} is invariant under $f^{2^{n-1}}$. A key idea in our argument is, for $\sqrt{2} < \sigma^{2^n} \leq 2$, to consider how the intervals in all of the sets $\Lambda_0, \Lambda_1, \dots, \Lambda_n$ fit together. Because we need to refer concurrently to the constituent intervals of several of these sets, we introduce an extra superscript into our notation, henceforth writing $I_i^{(n)+}$ to denote the interval I_i^+ from the set Λ_n . Our first proposition describes how the intervals relate between the sets Λ_{n-1} and Λ_n .

Proposition A.15

The interval $I_{2^n-1}^{(n-1)+}$ contains the two intervals $I_{2^n-1+r}^{(n)-}$ and $I_{2^n-1}^{(n)+}$, and no other interval $I_i^{(n)\pm}$. These two intervals are positioned so that the left end-point of $I_{2^n-1+r}^{(n)-}$ and the right end-point of $I_{2^n-1}^{(n)+}$ coincide with the left and right end-points of $I_{2^n-1}^{(n-1)+}$ respectively. When $\sigma^{2^n} = 2$ the intervals $I_{2^n-1+r}^{(n)-}$ and $I_{2^n-1}^{(n)+}$ touch, for all $\sigma^{2^n} < 2$ they are disjoint.

The relationships between the intervals of Λ_{n-1} and Λ_n implied by this proposition may be displayed diagrammatically as:

$$\frac{I_{2^n-1+r}^{(n)-}}{I_{2^n-1}^{(n)+}}$$

An analogous result shows that $I_{2^n-1}^{(n-1)-}$ contains the two intervals $I_{2^n-1}^{(n)-}$ and $I_{2^n-1+r}^{(n)+}$. We remark that Proposition A.15 strengthens the earlier result, Proposition A.14, since we know now that the intervals in Λ_n are disjoint for all $\sqrt{2} < \sigma^{2^n} < 2$.

Looking at the rightmost pair of intervals in Λ_n , $I_{2^n-1}^{(n)-}$ and $I_{2^n-1}^{(n)+}$, we can use our knowledge of the function f^{2^n} in this region to show that there is a periodic point located between them.

Proposition A.16

For $\sigma^{2^n} < 2$ there is a periodic point p_0 of least period 2^n , for f located between the adjacent intervals $I_{2^n-1}^{(n)-}$ and $I_{2^n-1}^{(n)+}$.

That p_0 , a fixed point of f^{2^n} , has least period 2^n , follows because the period is necessar-

ily a power of 2, and because the region between $I_{2^n-1}^{(n)-}$ and $I_{2^n-1}^{(n)+}$ lies beyond the outermost discontinuity of each f^i for $i < n$. We now consider points to either side of p_0 : they move away from p_0 under iteration by f because p_0 is repelling (recall that $a > 1$). p_0 divides the region between $I_{2^n-1}^{(n)-}$ and $I_{2^n-1}^{(n)+}$ into two parts which have to be treated separately.



Proposition A.17

(i) (Right Part) All points x with $p_0 < x < 1 - |I_{2^n-1}^{(n)+}|$ enter $I_{2^n-1}^{(n)+}$ after a finite number of iterations under f .

(ii) (Left Part) All points x with $1 - \sigma^{2^n}(1 - d_{n-1}) + |I_{2^n-1}^{(n)-}| < x < p_0$ enter $I_{2^n-1}^{(n)-}$ after a finite number of iterations under f .

To establish statement (i) we use that f^{2^n} increases linearly between p_0 and $I_{2^n-1}^{(n)+}$, with constant slope greater than 1. Statement (ii) initially appears more troublesome, because the effect of iterating f^{2^n} on a point x could be to map it into a region of $[-1, 1]$ where we do not know the form of f^{2^n} . The problem is overcome by first applying $f^{2^{n-1}}$ to x ; the point $f^{2^{n-1}}(x)$ is located near the left-hand end of $[-1, 1]$, to the left of the periodic point between $I_{2^n-1}^{(n)-}$ and $I_{2^n-1}^{(n)+}$, and we can argue exactly as in the proof of statement (i).

Using the definitions of the other intervals in Λ_n as the images under powers of f of $I_{2^n-1}^{(n)+}$ or $I_{2^n-1}^{(n)-}$, we can generalise Propositions A.16 and A.17 to other pairs of adjacent intervals in Λ_n .

Proposition A.18

For $\sigma^{2^n} < 2$ there is a periodic point of least period 2^n between $I_{2^n-1+r}^{(n)-}$ and $I_{2^n-1+r}^{(n)+}$ (and likewise between $I_i^{(n)-}$ and $I_i^{(n)+}$). All other points between these two intervals enter Λ_n after a finite number of iterations under f .

As stated, Proposition A.18 does not directly account for the orbits of all the points in the set $[-1, 1] \setminus \Lambda_n$: it only describes the orbits of points in the 2^n gaps between a pair of intervals $I_{2^{n-1},r}^{(0)-}$ and $I_{2^{n-1},r}^{(0)+}$, or $I_{2^{n-1},r}^{(1)-}$ and $I_{2^{n-1},r}^{(1)+}$, whereas there are in total $2^{n+1} - 1$ gaps between adjacent intervals in Λ_n . However we can apply it to the remainder of the gaps; the key point here is that for a given value of a , Proposition A.18 holds for all n such that $a^{2^n} < 2$. This, combined with Proposition A.15 on the nested structure of the intervals within the sets Λ_n , allows us to describe the effect of iterating a point from any of the gaps, including those not directly accounted for by Proposition A.18. The following instance illustrates the techniques used.

We suppose that $a^2 < 2$ and consider the set Λ_1 , with its four intervals $I_0^{(1)+}$, $I_0^{(1)-}$, $I_1^{(1)+}$ and $I_1^{(1)-}$. The diagram below shows these, in relation to the two intervals $I_0^{(0)+}$ and $I_0^{(0)-}$ of Λ_0 .

$$\begin{array}{c} I_0^{(0)-} \quad I_0^{(0)+} \\ \hline I_0^{(0)-} \quad I_1^{(1)+} \quad I_1^{(1)-} \quad I_0^{(1)+} \\ \hline q_1 \quad p_0 \quad q_0 \end{array} \quad \begin{array}{c} \Lambda_0 \\ \hline \Lambda_1 \end{array}$$

The behaviour of points in the gaps between $I_0^{(1)-}$ and $I_1^{(1)+}$, and $I_1^{(1)-}$ and $I_0^{(1)+}$, follows directly from Proposition A.18: apart from the period-2 orbit $\{q_0, q_1\}$, all points enter Λ_1 after a finite number of iterations. The remaining gap, between $I_1^{(1)+}$ and $I_1^{(1)-}$, coincides exactly with the gap between $I_0^{(0)-}$ and $I_0^{(0)+}$, and we invoke Proposition A.18 with $n = 0$, to assert that the orbit of a point x from this region is either the fixed point p_0 , or enters Λ_0 after a finite number of iterations. Once in Λ_0 , the iterate of x is either also in Λ_1 , or else in one of the two gaps already dealt with. Thus we can summarise the various possibilities for the iteration of a point in $[-1, 1] \setminus \Lambda_1$: iterates coincide with the fixed point p_0 or the 2-cycle $\{q_0, q_1\}$; iterates start in $[-1, 1] \setminus \Lambda_0$ and eventually coincide with the 2-cycle $\{q_0, q_1\}$; iterates eventually enter the set Λ_1 on which f is topologically transitive.

The ideas used here generalise to the following result, establishing that for those parameter values for which we have shown that f is topologically transitive on Λ_n , namely $\sqrt{2} < a^{2^n} \leq 2$, Λ_n is an attracting set.

Starting with $R_0^{(0)+}$, i.e. the left end-point of $I_0^{(0)+}$, given by $1 - a^{2^n}(1 - d_n)$, we can show that all these functions are polynomials.

Proposition A.20

The functions $L_r^{(n)+}$ and $R_r^{(n)+}$ are polynomials in a .

It is of interest to know the dependence on the parameter a of both the gap width $w_r^{(n)} = R_r^{(n)+} - L_r^{(n)+}$ between the two intervals $I_{2^{n-1},r}^{(n)-}$ and $I_{2^{n-1},r}^{(n)+}$, and also the proportion $p_r^{(n)} = w_r^{(n)} / |I_r^{(n-1)+}|$ of the containing interval $I_r^{(n-1)+}$ occupied by this gap.

Proposition A.21

$$(i) \quad w_0^{(n)}(a) = 4 - 2a, \quad w_r^{(n)}(a) = a^r(2 - a^{2^n})(a - 1) \prod_{i=0}^{n-2} (a^{2^i} - 1) \quad \text{for } n \geq 1.$$

$$(ii) \quad p_r^{(n)}(a) = 2 - a^{2^n}$$

These quantities give information on the size of the wandering set $[-1, 1] \setminus \Lambda_n$ in comparison to the size of the attracting set Λ_n . We draw the following conclusions.

- The proportion of the interval $I_r^{(n-1)+}$ taken up by the gap between $I_{2^{n-1},r}^{(n)-}$ and $I_{2^{n-1},r}^{(n)+}$ increases monotonically from 0 at $a^{2^n} = 2$, approaching 1 as $a^{2^n} \rightarrow 1$. This means that as a decreases, the size of $\Lambda_{n(a)}$ becomes smaller and smaller in comparison to the size of the wandering set.
- For $n \geq 1$, the width of the gap between $I_{2^{n-1},r}^{(n)-}$ and $I_{2^{n-1},r}^{(n)+}$ is zero at $a^{2^n} = 2$ and again at $a = 1$. For all values of a between these two extremes, the hole has nonzero width. Note that this does not contradict (a) because the gap between $I_0^{(0)-}$ and $I_0^{(0)+} = [-1, a - 2]$ and $I_0^{(0)+} = [2 - a, 1]$ expands to fill the whole interval $[-1, 1]$ as a approaches 1, the intervals in $\Lambda_{n(a)}$ and all other gaps becoming increasingly squashed in the diminishing regions to either side of this central gap.
- $w_r^{(k)} < w_r^{(j)}$ whenever $k > j$, and this coupled with the work from Section A.2.2 on the nested structure of the intervals in Λ_n allows us to compute the relative sizes of the gaps between the intervals in Λ_n .

Proposition A.19

For $\sqrt{2} < a^{2^n} \leq 2$, the orbit under f of a point in $[-1, 1] \setminus \Lambda_n$ is of the three following types.

- A periodic point, of period 2^i for some $0 \leq i \leq n$.
- Eventually periodic: the iterates coincide with one of the points in case 1 after a finite number of iterations.
- The iterates enter Λ_n after a finite number of iterations.

Case 3 applies to (Lebesgue) almost every point in $[-1, 1]$.

A.2.3 Changes in the structure of Λ_n with the parameter a

We know from the previous section that as the parameter a decreases through the value $a^{2^n} = 2$, the behaviour, under iteration by f , of points within an interval $I_{2^{n-1},r}^{(n-1)+}$ changes. The interval splits into three parts: two intervals $I_{2^{n-1},r}^{(n)-}$ and $I_{2^{n-1},r}^{(n)+}$ on which f^{2^n} is topologically mixing, and a gap between these two intervals whose points, with the exception of one periodic point, enter the set Λ_n after a finite number of iterations. Suppose we plot Λ_n against a , that is create a plot showing Λ_0 for $\sqrt{2} < a \leq 2$, Λ_1 for $\sqrt{2} < a^2 \leq 2$, etc.: the resulting plot looks like Figure A.2(a). It will subsequently be convenient to refer to this as the plot of $\Lambda_{n(a)}$ against a . In it a hole appears where $I_{2^{n-1},r}^{(n-1)+}$ gives way to $I_{2^{n-1},r}^{(n)-}$ and $I_{2^{n-1},r}^{(n)+}$ as a is decreased through $a^{2^n} = 2$, and this persists as a is further reduced down to 1. Some of these holes can be seen in Figures A.2(a) and A.2(b), the three largest are the ones which commence at $a = 2$ (the large central triangular hole) and at $a = \sqrt{2}$ (the two holes either side of the triangular hole).

We denote by $L_r^{(n)+}$ and $R_r^{(n)+}$ the functions describing the bounding curves for the hole, in the plot of $\Lambda_{n(a)}$ against a , arising from the gap between the intervals $I_{2^{n-1},r}^{(n)-}$ and $I_{2^{n-1},r}^{(n)+}$. (There are of course corresponding functions $L_r^{(n)-}$ and $R_r^{(n)-}$ describing the location of the gap between intervals $I_r^{(n)-}$ and $I_{2^{n-1},r}^{(n)+}$; these are related to $L_r^{(n)+}$ and $R_r^{(n)+}$ via $L_r^{(n)-} = -R_r^{(n)+}$ and $R_r^{(n)-} = -L_r^{(n)+}$, so we will not refer to them again in this section.)

We develop point (b) further in the following result.

Proposition A.22

The polynomial $w_r^{(n)}$ has a unique maximum for $1 < a^{2^n} < 2$. This maximum is located in the range $\frac{1+a^2}{1+a} \leq a \leq \rho$, where $\rho = 2^{\frac{1}{2^n}}$.

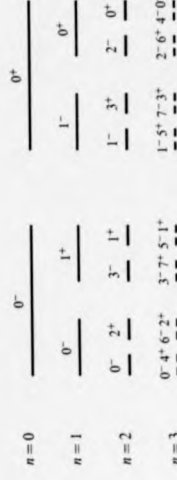
It follows that the holes in a plot of $\Lambda_{n(a)}$ against a increase in width, as a increases from 1, for a proportion of at least $\frac{1}{1+a}$ of their total height $\rho - 1$, and the maximum width occurs in the uppermost part when the height is divided into $n + 1$ equal segments. The numbers $\frac{a^n}{1+a}$ form the sequence $\frac{1}{2}, \frac{3}{4}, \frac{3}{4}, \dots$ so the holes corresponding to large n must assume their maximum width very near their peaks.

A.2.4 Examples

Many of the results and proofs of the previous sections are constructive in nature, and we include two examples here showing how the constructions give specific information about the dynamics of the map f and the structure of the set Λ_n .

Example A.1: Order of the intervals $I_r^{(n)\pm}$ within Λ_n

Proposition A.15, on the nesting of intervals, shows that as a decreases through $a^{2^n} = 2$ the interval $I_r^{(n-1)+}$ splits, giving rise to $I_{2^{n-1},r}^{(n)-}$ and $I_{2^{n-1},r}^{(n)+}$. Commencing with $n = 0$, corresponding to parameter values $\sqrt{2} < a \leq 2$, this result can be used inductively to establish the order of the 2^{n+1} intervals, as they occur along the line segment $[-1, 1]$, within the set Λ_n . The diagram below illustrates this for $n = 0, 1, 2, 3$; for clarity we typically denote the interval $I_r^{(n)+}$ simply by r^+ .



The recondite condition in our **while** loop implementing step 2 merits further explanation. Within the loop j is advanced while $x = h(\text{table}[j])$ corresponds to a position beyond i . In the usual situation, when the end of the table does not come between positions i and j , the condition to remain within the loop is simply $i < x \leq j$. When the end of the table intervenes, so that $j < i$, then we remain within the loop so long as x is between position i and the end of the table (i.e. $i < x$) or x is between the start of the table and the position j (i.e. $x \leq j$). The following diagrams illustrate these three situations and associated with each is the conditional expression that characterises it.

usual situation $\begin{array}{c} x \\ \text{---} \\ i \quad j \end{array}$ $i < x \ \&\& \ x \leq j$

situations involving
wrap-around $\begin{array}{c} x \\ \text{---} \\ i \quad j \end{array}$ $i < x \ \&\& \ j < i$

$\begin{array}{c} x \\ \text{---} \\ i \quad j \end{array}$ $x \leq j \ \&\& \ j < i$

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